On the Absence of Rate Loss in Decentralized Sensor and Controller Structure for Asymptotic Stability

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***Abstract*— We consider a noiseless multi-sensor LTI system where the initial state has a continuous density over a bounded set and the channels connecting the sensors to the controller are discrete and noiseless. We study the rate requirements for the stability of such a system. We show that when the overall system is stabilizable and detectable, the rate required for asymptotic stability is at most an arbitrarily small $\epsilon > 0$ larger than $\sum_{|\lambda_i|>1} \log_2(|\lambda_i|)$, where λ_i's are the open-loop eigenvalues of the system. Thus, there is no loss in performance due to decentralization. We provide a sequential encoding scheme achieving this rate.**

I. Introduction

A. Problem formulation

We consider a remotely controlled n-dimensional discrete-time LTI system with the dynamics

$$x_{t+1} = Ax_t + Bu_t\ ,\ t \geq 0, \tag{1}$$

where (A, B) is stabilizable, x_t is the state, u_t is the control, and the initial state x_0 is a random vector with a known continuous distribution over a compact support.

There are L sensors with the mth one observing $y^m = C^m x, 1 \leq m \leq L$, where $y^m \in \mathrm{R}^{l_i}$ (see Fig. 1). We assume $(A, [(C^1)^T \dots (C^L)^T]^T)$ to be detectable and that A is in Jordan form $A = (J^u, J^s)$, where J^u is a $k \times k$ matrix with unstable eigenvalues and J^s is a $n-k \times n-k$ matrix with stable eigenvalues.

In our setup, the communication channels connecting the sensors to the controller are discrete and noiseless. We denote the encoder output at sensor i transmitted to controller at time t by z_t^i. The sensors have access to the following:

$$I^i_{\text{sensor},t} = \{u_{[0,t-1]}, z^i_{[0,t-1]}, y^i_t\}$$

The controller is a centralized one, and has access to all the sensor outputs:

$$I_{\text{controller},t} = \{u_{[0,t-1]}, z_{[0,t]}, \hat{x}_{[0,t-1]}\}$$

We consider fixed-rate quantizers, i.e., the rate is defined to be the logarithm of the number of symbols transmitted over the discrete channel. Let $R^i, 1 \leq i \leq L$, be the rates of the encoding outputs of the sensors. Throughout, we have $z_t^i = \gamma_t^i(I^i_{\cdot,\text{sensor},t}) \in \{1, 2, \dots, 2^{R^i}\}$, for some sensor functions (encoders) γ_t^i. We seek a solution to the following problem.
Problem: Let $\mathcal{R}_\mathcal{A}$ be defined as

$$\mathcal{R}_\mathcal{A} = \{R^1, R^2, \dots, R^L : \exists u_{[1,2,\dots,\infty)}, \lim_{T\to\infty} x_T = 0; \},$$

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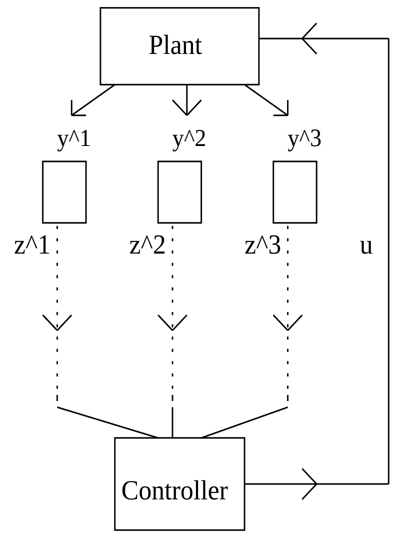

Fig. 1: Multi-sensor system structure

We seek the minimum such rate in the sense that

$$R^* = \inf_{R^i \in \mathcal{R}_\mathcal{A}} \sum_{i=1}^{L} R^i,$$

and an encoding scheme that achieves this rate. ◇

B. Relevant literature

Control of physically distant systems over communication channels has recently emerged as a major research topic. Recently, this analysis has included decentralized and multi-sensored systems as well. In decentralized systems, depending on the availability of information transfer among subsystems, different decentralized observer and controller designs are possible has been as observed in [1], [2], [3]. However, these references have not considered information issues. Some representative papers specifically focusing on information constraints are [9], [6], [4], [11] and [8]. The first one, [9], studies the remote control problem from a resource allocation point of view, and obtains achievable rates via linear matrix inequalities. Reference [7] uses binning schemes in a decentralized control context. Reference [11] carries out a sufficient rate analysis in a multi-controller setting, where the open-loop state dynamics are decoupled, and *multi*-controllers act upon different plants with limited information available to them from various plants. Reference [6] primarily studies the case where the modes observable by the sensors are decoupled. Such a restriction reduces the problem to a number of centralized encoding problems. The most general case in which the observable modes are not separable, that is, the observation vectors at each of the sensors are not sufficient to extract each of the observable modes themselves, or the case in which the observable modes at each of the sensors are coupled have not been studied. Nonetheless, reference [6] comments on a special case where the assumption on separability does not hold, and studies the case where the sensors observing the same modes cooperate while encoding the same mode.

The results closest to ours have been presented in [4] and [5], which provide a comprehensive treatment of distributed control with communication constraints and necessary and sufficient conditions on stability. Indeed, references [4], [5] are the first works that have argued that the minimum rate required is the same as the rate required in the centralized case. This current paper is different from [4], [5] in that we make the connection with Slepian-Wolf coding theorem precise since Slepian-Wolf coding is only valid for discrete sources, and further we explicitly construct encoding schemes using both geometric and analytic tools as we had done earlier in [7] and [8]. Reference [8] studies the case when there is bounded noise in the system, and hence Wyner-Ziv [17] coding arguments and binning schemes are used to obtain sufficient rate regions leading to stability. In this paper, however, there is no noise and this enables us to achieve a precise result. Our analysis in this paper builds on the correlation of the encoded data by the sensors who do not communicate among themselves.

More formally, we prove in this paper that there is no loss in rate performance due to decentralization, when the objective is asymptotic stability. We provide a sequential coding scheme based on the QR decomposition achieving the rate bound.

C. Preliminaries

A *quantizer*, Q, for a continuous variable is a mapping from the real line to a finite or countable set, characterized by corresponding bins $\{\mathcal{B}_i\}$ and their reconstruction levels q^i, such that $\forall i$, $Q(x) = q^i$ if $x \in \mathcal{B}_i$ and $q^i \in \mathcal{B}_i$.

We define a mode of a linear system as the eigenvectors of the system matrix A. Note that in this definition, some modes can be coupled since there may be generalized eigenvectors, and hence may belong to the same Jordan block. The state space can be expressed as a superposition of modes.

We define $o_m(y^m)$ as the set of unstable modes in the observable space of sensor m, whose initial states are recovered. We let x^u denote the unstable modes of the system and x^s denote the stable ones. We say an unstable mode, i, is separable if, x_0^i is recoverable by at least one sensor; i.e., $x_0^i \in o_m(y^m)$, for some m; the state is separable if $x_0^u = \bigcup_{1\le m\le L} o_m(y^m)$. We say two modes are decoupled, if the open loop dynamics of them are independent, otherwise they are coupled. This classification will be helpful in the development. Throughout, for two vectors u, v, the notation $u \le v$ denotes componentwise inequality, i.e., $u^i \le v^i \forall i$.

Before proceeding further, let us comment on the effect of the stable modes. For the stable modes, there does not need to be any communication, since these modes asymptotically go to zero, and they are asymptotically ineffective in the evolution of the observation outputs, since $\lim_{t\to\infty} A_s^t x^s = 0$. Thus, we have

$$\lim_{t\to\infty} ||A^t x_0^u - A^t x_0||_\infty = 0.$$

Due to space limitations, we omit the technical discussion played by the stable modes in the following analysis; we include this in a longer version.

We denote by $G = (O^u)^{-1}$ the transformation matrix to recover the unstable modes of the system, such that $\hat{x}_0^u = GY$, with the assumption that stable modes are all zero. We have $Y = [(Y^1)^T \quad (Y^2)^T \quad \dots \quad (Y^L)^T]^T$, where the upper case letters, such as

$$Y^i \quad = \quad \left[(C^{i,u})^T (C^{i,u} A^u)^T \dots (C^{i,u}(A^u)^{r-1})^T\right]^T x_0^u$$

represent the sufficient observation vectors needed for the recovery of the unstable modes of the initial state, each provided by the corresponding sensors. Let $y^i \in \mathrm{R}^{\mathrm{l_i}}$. Then, $C^{i,u} : \mathrm{R}^{\mathrm{k}} \to \mathrm{R}^{\mathrm{l_i}}$ denotes the observation matrix, mapping the the unstable subspace to the observation range of sensor i. We let $x_{[0,t]}$ denote the set of all state vectors up to time t, $\{x_0, \dots, x_t\}$, and likewise $u_{[0,t]}$ denote the set of all control vectors up to time t, $\{u_0, \dots, u_t\}$. Superscript i will designate the i-th component of that sequence, which is not necessarily scalar, so that, for example, $x^i_{[0,t]} = \{x^i_0, \dots, x^i_t\}$. In each case, $I^i_{\cdot,\text{sensor},t}$ will denote the information available to sensor i at time t.

Since control is assumed to be available to the sensors, there is no *dual effect* of control, i.e., control does not impact the state estimation error. Therefore, to compute the minimum rate requirements, without any loss of generality, we assume the system to be control-free. Later on, however, we will consider control in the final construction and describe how the state can be designed to converge to zero asymptotically.

D. Converse bound

As a lower bound to the rate problem introduced, one can consider a centralized system, with a single observation post. The lower bound for such a centralized system for asymptotic stability is given by

$$\sum_{|\lambda_i|>1} \log_2(|\lambda_i|) + \epsilon,$$

for some arbitrarily small $\epsilon > 0$ (see [5] and the references therein). Since the decentralized encoder has fewer degrees of freedom for encoding, this rate value is a lower bound for any achievable decentralized encoding scheme.

The organization of the rest of the paper is as follows. We first provide a discussion on distributed coding of correlated sources in section II, first in a broader context and then more specifically in the context of a linear source. Following this, we obtain achievable rates and constructions for the multi-sensor case in section III. We discuss the multi-controller setup in section IV. The paper ends with the concluding remarks of section V.

II. Coding for Correlated Sources

A. Decentralized coding for correlated sources: Slepian-Wolf and Wyner-Ziv coding theories

Let X and Y be two correlated discrete-valued random variables. Slepian and Wolf [15] showed that, to be able to perfectly decode both signals (so that the estimation error is zero), the achievable rate region is: $\{R_x, R_y : R_x \geq H(X|Y), R_Y \geq H(Y|X), R_x + R_y \geq H(X,Y)\}$. Thus there is no loss in encoding separately, if it is desired to minimize the sum of the rates. This result extends to more than two sources as well. Wyner and Ziv [17] studied the case where the reconstruction is not perfect. Their analysis has been extended in the literature for multi-terminal encoding systems (see for example [13], [20]). Wyner [16] extended the analysis to continuous sources. In general, when the recovery is not perfect, that is the distortion is non-zero, there is a strict rate loss in the Wyner-Ziv problem as opposed to the Slepian-Wolf problem. However, when the distortion goes to zero, then the rate loss of distributed coding is zero [18] even for the continuous alphabet, smooth sources. Thus, there exists a Wyner-Ziv coder which is as good as the best one which has perfect access to the correlated random variable.

For any finite number of sensors, the achievable rate region in the Slepian-Wolf setting is precisely known. In a direct application of the Slepian-Wolf coding theorem to our problem, however, there is a technical difficulty that needs to be overcome. In our analysis, we have multiple, possibly more than two, sensors, and the source to be encoded has a continuous distribution, which eliminates a direct use of Slepian-Wolf coding. We consider in this paper, however, the case where the estimation error converges to zero. Therefore, if the continuous source can be transformed to a discrete source as a result of fine quantization, one can carry on the analysis with discrete sources. Furthermore, since we assume our source has a compact support, such a transformation is not difficult; using high-rate uniform quantization ensures a smooth transformation, that will be made precise in the next section.

We now illustrate distributed coding gain, with an example in a linear systems context [8]. This illustration will be helpful in the analysis of the next section as well. A typical evolution in a linear system starting from the unit box is given in the following figures. At each stage the sensors have access to the uncertainty polygon that they are transmitting (see Figure 2). Thus, each sensor is capable of considering the support sizes of the errors at all the controllers, and this can be exploited in reducing the data rates (Figs. 3 and 4).

In the following, we study different cases.

1) Case 1: Separable, decoupled modes: The unstable modes, $o_m(y^m)$ that are observable by sensor m satisfies $\bigcup_m o_m(y^m) = x^u$, where x^u denotes the unstable modes, and $o_m(y^m)$ are decoupled. The unstable modes of the initial state $o_m(y^m)$ can be recovered in $T_m = p_m$ stages, where p_m is the dimension of the unstable subspace observed by sensor m. Let $T := \max_{1 \leq m \leq L}\{T_m\}$. All the observable modes will be recovered by time T.

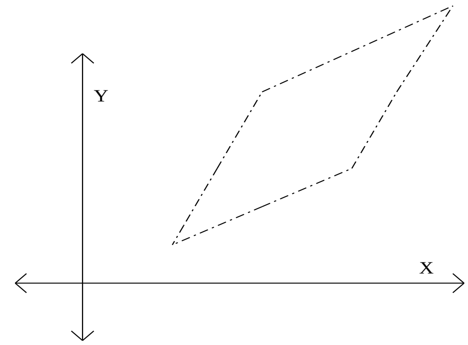

Fig. 2: The symbols to be encoded are correlated.

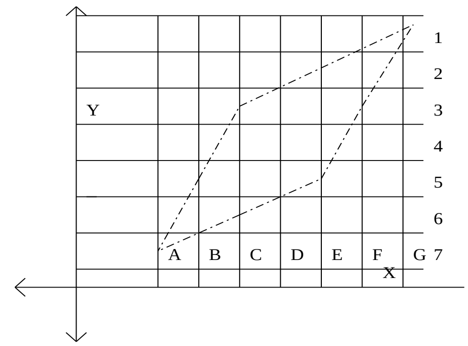

Fig. 3: If the sensors do not collaborate then each of their corresponding sensors will send information for 7 symbols.

This case arises in a system of the following type:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},$$

$$y^1 = [1 \quad 0]x; \quad y^2 = [0 \quad 1]x.$$

Since any of the observable sets is decoupled from the rest, the problem is one of a number of independent centralized encoding problems; each system can be assumed to be centralized within itself and the optimal schemes can be computed using known results in the literature. This case has been studied in [6].

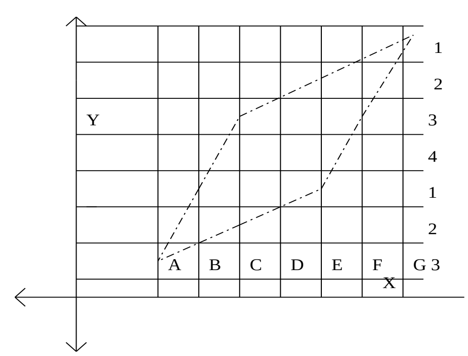

Fig. 4: For any level that sensor X has to send, there are only 4 levels, and not 7, that sensor Y needs to send. Thus there is a gain in encoding.

2) Case 2: Coupled, separable modes: In this scheme, as was in case 1, we also have $\bigcup_m o_m(y^m) = x^u$, where x^u denotes the unstable modes, but here the modes are coupled.

Thus, after some time $T \leq n$, all sensors have access to what they need to transmit. The only case in which coupling might occur is due to the Jordan form. Such a case arises in a system of the following type:

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix},$$

$$y^1 = [1 \quad 0]x; \quad y^2 = [0 \quad 1]x.$$

Suppose we encode each of the modes with R_i bits per stage. Since the growth in the off-diagonal terms is linear in time, stability is characterized by the eigenvalues of each of the modes. Therefore the rate needed is $R = \sum_{|\lambda_i|>1} \log_2(|\lambda_i|) + \epsilon$, for some arbitrarily small ϵ.

3) Case 3: Inseparable modes: This is the case where the sensors can not recover the modes themselves, and the controller needs a joint decoding to observe the output. Such an example would be the following.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

$$y^1 = [1 \quad 0 \quad 0]x; \quad y^2 = [0 \quad 1 \quad 1]x.$$

Note that neither of the pairs $(A, C^1), (A, C^2)$ is detectable. We study case 3 in the remainder of the paper, for this is the most general case.

III. Stabilizing Rates and Rate Loss

Since the initial condition has a continuous density function with a compact support, we can use the Lemma 3.1 of [12] or a special case of an argument in [19] which would reduce to the following: Let an $n-$dimensional vector random variable X have a finite differential entropy and have a continuous distribution function over a compact support set. Let X be quantized uniformly with n-dimensional unit cubes, with bin sizes δ. Then the following relationship holds between the entropy of the quantized state and the entropy of the state:

$$\lim_{\delta \to 0} \Big(H(Q(X)) + n \log_2(\delta) \Big) = H(X).$$

In view of this, we will also have

$$\lim_{\delta \to 0} \Big(H(Q(AX)) + n \log_2(\delta) \Big) = H(AX).$$

Hence, in the fine quantization limit,

$$\lim_{\delta \to 0} \Big(H(Q(AX)) - H(Q(X)) \Big) = \log_2(|A|)$$

Thus the additive evolution of the entropy is not lost due to fine quantization. As t becomes large, for a sufficiently small δ_0, and a small ϵ, we will have

$$\lim_{t \to \infty} 1/t |H(Q(A^t X)) - (H(A^t X) - n \log_2(\delta_0) + \epsilon)| = 0.$$

Thus, using continuous sources which are finely quantized, we can study discrete valued sources without affecting the linear evolution.

A. Two sensor case

Define

$$\mathcal{R}_{\mathcal{A}} = \{R^1, R^2 : \exists u_{[1,2,\ldots,\infty)}, \lim_{T \to \infty} x_T = 0; \}.$$

$$R_{\text{loss}} = \inf_{\mathcal{R}_{\mathcal{A}}} \sum_1^2 R^i - \sum_{|\lambda_i|>1} (\log_2(|\lambda_i|)).$$

Theorem 3.1: For the decentralized system (1) with two sensors the rate loss is zero. Thus there is no loss of optimality in separate encoding by sensors.

Proof: Suppose that $x_0 = G[y^{1T} \quad y^{2T}]^T$. Then, at time t, we have $A^t G[y^{1T} \quad y^{2T}]^T$. Here we assume the y vectors to be discrete valued. In the Slepian-Wolf coding, $H(y^2, H(A^t G[y^1 \quad y^2]^T|y^2)$ is a corner point of the achievable rate region. But we know that $H(y^1) + H(A^t G[y^1 \quad y^2]^T|y^1) = H(A^T G[y^1 \quad y^2]^T)$. Let K be the differential entropy of the unquantized state. Thus, using the arguments in [12], the entropy satisfies:

$$\begin{aligned} & \lim_{t \to \infty} 1/t(H(A^t x_0)) \\ = \quad & \lim_{t \to \infty} 1/t(\log_2(|A|) + \log_2(|G|) + K - \lim_{\delta \to 0} 2\log(\delta)) \\ = \quad & \lim_{t \to \infty} 1/t(\log_2(|A|) + \log_2(|G|) + K + 2\log(t)) \\ \leq \quad & \log_2(|A|) + \epsilon \end{aligned} \tag{2}$$

for some arbitrary $\epsilon > 0$. Note that, here the speed of decay of δ to zero can be picked to be slower than the speed of decay of $1/t$. Thus, the rate $\sum_1^2 \log_2(|\lambda_i|) + \epsilon$ for some arbitrarily small $\epsilon > 0$ is sufficient. ◇

A Geometric Proof We now provide a geometric proof. The proof is for the case when the modes are orthogonal, but it can be extended to the general case. Consider Fig. 5. We observe that the evolution is parallel to the eigenvectors (in the coordinate system defined by the observation vector). As we observed in Fig. 4, the lengths of the strips h_1 and h_2 are what determine the number of bins to be encoded. Using a similar argument now for infinitesimal widths in the bins in either modes, we need to encode h_1 and h_2.

We will prove that $\lim_{t \to \infty} h_1 h_2 / |\lambda_1 \lambda_2|^t = \eta$, where η is a constant. Suppose that at time $t = 0$, the lengths of both of the diagonals are 1. Consider the angles θ and α. Note that, $\tan(\alpha) = (|\lambda_2|/|\lambda_1|)^t$. We have,

$$h_1 = \sqrt{|\lambda_1^{2t} + \lambda_2^{2t}|}(\sin(\theta + \alpha) - \cos(\theta + \alpha)\tan(\theta - \alpha))$$

$$h_2 = |\lambda_1^t| \cos(\theta)$$

$$\begin{aligned} h_1 h_2 = & \sqrt{\lambda_1^{2t} + \lambda_2^{2t}} (\lambda_1^t) \Big(\cos(\theta) / \cos(\theta - \alpha) \Big) \\ & . \Big(\sin(\theta + \alpha)\cos(\theta - \alpha) - \cos(\theta + \alpha)\sin(\theta - \alpha) \Big) \end{aligned} \tag{3}$$

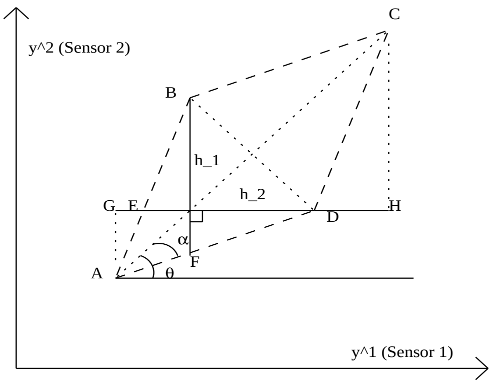

Fig. 5: $\vec{AC}$ and $\vec{DB}$ are the orthogonal scaled eigenvectors. For a given value of y^1, the uncertainty is within a strip of length $h_1 = |\vec{BF}|$. Likewise, for any given y^2 value, the uncertainty is within an interval of $h_2 = |\vec{GH}|$. The product of h_1 and h_2 is a scalar multiple of the length of the eigenvectors for small α.

After a few steps involving trigonometric identities, we obtain:

$$h_1 h_2 = \frac{2(|\lambda_1|^t |\lambda_2|^t)\cos(\theta)}{\cos(\theta-\alpha)/\cos(\alpha).}$$

Suppose $\lambda_2 < \lambda_1$. Then, we have $\lim_{t\to\infty} \alpha = 0$. Since

$$\lim_{\alpha\to 0} \cos(\theta-\alpha)/\cos(\alpha) = 1,$$

and θ is a constant, we have $\lim_{t\to\infty} h_1 h_2 = 2|\lambda_1|^t|\lambda_2|^t$. Thus, we have $\eta = 2$. The only case not studied above is when $\lambda_1 = \lambda_2$. In this case the polyhedron will be a square and simple geometric analysis shows that $\eta = 2$ holds in this case as well. ◇

B. Multi-sensor case

For the general multi-sensor case, we will once again consider the continuous sources to be masses of infinitesimal hyper-cubes. Therefore, multi-source Slepian-Wolf coding would be applicable. We state the main theorem of the paper.

Theorem 3.2: Define

$$\mathcal{R}_{\mathcal{A}} = \{R^1, R^2, \ldots, R^L : \exists u_{[1,\infty)}; \lim_{T\to\infty} x_T = 0\},$$

$$R_{\text{loss}} := \inf_{\mathcal{R}_{\mathcal{A}}} \sum_i R^i - \sum_{|\lambda_i|>1} (\log_2(|\lambda_i|)).$$

For the decentralized system (1), the rate loss is zero. Thus there is no loss in performance due to separate encoding by sensors.

Proof: Suppose $x_0 = G[y^1 \quad y^2 \quad \ldots \quad y^L]^T$, where G is the transformation matrix as introduced earlier.

We apply Slepian-Wolf Coding theorem for L sources ([14], pg. 415), for which the corner points of the convex achievable rate region are well known. For the extension to multi-sensor case, we will use the QR decomposition. For any real-matrix V, there exists an upper triangular matrix R and an orthonormal matrix Q such that, $VG = QR$. In our case, we have $A^tG = QR$. Note that with such a transformation the eigenvalues are not necessarily preserved, but since $|A^tG| = |A|^t|G| = |Q||R| = |R|$, the determinant of A^tG is identical to that of the matrix R. Thus, $Q^{-1}x_t = RY$. Hence, we can achieve the following corner point in the Slepian-Wolf achievable rate region:

$$\begin{aligned}&(H(R_{L,L}Y^L),\\ &H(R_{L-1,L-1}Y^{L-1} + R_{L-1,L}Y^L|Y^L), \ldots,\\ &H(R_{1,1}Y^1 + R_{1,2}Y^2 + \cdots + R_{1,L}Y^L|Y^1, \ldots, Y^{L-1}))\end{aligned}$$

Note that here the sum rate equals $H(RY)$. Since Q is orthonormal, the multiplication by Q does not alter the entropy. We thus have,

$$H(Ry) = t\log_2(|R|) + H(Y).$$

Since this equals to $t\log_2(|A|) + \log_2(|G|) + H(Y_1^L)$, and

$$H(Y_1^L) = \lim_{\delta\to 0} K - k\log_2(\delta),$$

where K is the finite differential entropy of the unquantized vector and k is the dimension on Y_1^L, we have

$$\begin{aligned}&\lim_{t\to\infty} 1/t\{t\log_2(|A|) + \log_2(|G|) + H(Y_1^L)\}\\ &= \lim_{t\to\infty} 1/t \lim_{\delta\to 0}\{t\log_2(|A|) + \log_2(|G|) + K - k\log_2(\delta)\}\end{aligned}$$

Since the goal is asymptotic stability in the sense that $\lim_{t\to\infty}\delta = 0$, the speed of convergence of the bin size can be picked to satisfy the following:

$$\begin{aligned}&\lim_{t\to\infty}\lim_{\delta\to 0} 1/t\{t\log_2(|A|) + \log_2(|G|) + K - k\log_2(\delta)\}\\ &= \lim_{t\to\infty} 1/t\{t\log_2(|A|) + \log_2(|G|) + K + k\log_2(t)\}, \quad (4)\end{aligned}$$

which is arbitrarily close to $\log_2(|A|)$.

Construction:

We first define

$$R_t := 1/t\{t\log_2(|A|) + \log_2(|G|) + K + k\log_2(t)\}.$$

Let the target time for first series of encodings be $T_t > 0$. From the analysis above, for any $\epsilon > 0$ and $\delta > 0$, $\exists$ $t = T_t > 1$ such that R_t is less than $\log_2(|A|) + \epsilon$. Pick $\delta_0 = ||x_0 - \hat{x}_0||_\infty$, where $\hat{x}_0$ is the mid-point of the uncertainty region (bin) where the initial state resides. Let $\delta < \delta_0$ and pick an arbitrarily small ϵ. Find an appropriate $t = T_t$ such that the above holds. Now, let $A^tG = QR$ as above. Let $RY = v$, where v is a vector with the same dimension as Y. Note that the corner point above is now known to be achievable from the analytical analysis above. Our construction builds on this corner point whose sum-rate achieves the entropy bound. First, the Lth sensor encodes v^L. Given v^L, the controller can extract Y^L through the following relation:

$$Y^L = R_{L,L}^{-1} v^L.$$

Then the $L-1$th sensor encodes v^{L-1} conditioned on the value of Y^L (as was done in the two sensor case). Given this value, Y^{L-1} is recovered by the controller:

$$Y^{L-1} = R_{L-1,L-1}^{-1}(v^{L-1} - R_{L-1,L}Y^L).$$

Thus, all $v^m, \quad m \geq 1$, values are encoded by the corresponding sensors conditioned on the values of Y^i for all $i > m$.

At the controller, Y^m values are recoverable, given v^m and $Y^k, k > m$:

$$Y^m = R_{m,m}^{-1}(v^m - \sum_{j>m}^{L} R_{m,j} Y^j).$$

This recovery is possible due to the upper diagonal structure of the matrix R.

Such a sequential decoding lets us achieve the Slepian-Wolf bound. The rate needed for this scheme is given by the diagonal terms of the matrix R. Thus at the end of T_t seconds we have obtained a smaller uncertainty region.

Now apply the same scheme starting from time $t = T_t + 1$ until $t = 2T_t$, replacing δ_0 with δ. Such a recursion will ensure that δ values are decreasing geometrically and asymptotic stability will be obtained. The rate needed is $\log_2(|A|) + \epsilon$, where $\epsilon > 0$ is arbitrarily small. Note that this is identical to the condition in the centralized case, and hence, this rate is both necessary and sufficient; and thus the bound is tight.

The controller uses its received data, and takes the unstable modes in its estimate to zero in at most r time stages (this is possible due to the controllability assumption for the unstable modes)

$$u_{kT_t} = f(\hat{x}_{kT_t}),$$

where $\hat{x}$ is the estimate of the state, so that at times $kT_t, k \geq 0$, the state uncertainty region gets closer to the origin, and thus $\lim_{t\to\infty} x_t = 0$. $\diamond$

IV. Multi-Control Systems

The analysis in the multi-sensor case can be extended to multi-controller systems. As long as there is a centralized decoder, it tuns out that the same rate is necessary and sufficient for stabilizability, since in the final analysis what matters is the availability of observer information. However, the union of the modes in the system that are controllable and observable has to span the entire unstable space for controllability; in other words for each unstable mode, there has to be at least one subset of observer and controllers that can jointly detect and stabilize the modes. The proof again follows the insight given by the Slepian-Wolf coding theorem, with the assumption that the plant can jointly decode the received messages.

V. Concluding Remarks

In this paper, we showed that there is no rate loss due to sensors' partial access to the source; furthermore there exists a coding scheme achieving a rate of $\sum_{|\lambda_i|\geq 1} \log_2 |\lambda_i| + \epsilon$, where ϵ is arbitrarily small.

The reason there is no rate-loss is because the recovery is perfect. If there were noise in the system, then the recovery would not necessarily be perfect and there would be a strict rate loss unless the decentralized sensors observe decoupled plants. The case where the channels are noisy is currently under study. We note that distributed coding through Gaussian channels has extensively been studied in the communications literature (see [20]), in the context of what is also known as the Gaussian CEO (Chief Executive Officer) problem. However, the issues with regard to feedback and control have not been investigated, and remain as topics for future research.

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