Optimal Tracking with Feedback-Feedforward Control Separation over a Network

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***Abstract*— This paper studies tracking of a reference path in a networked control system where the controller consists of a central decision maker and an on-site controller, which are connected through a discrete noiseless channel. The reference path is available noncausally to the central decision maker and the on-site controller has access to noisy observations from the plant and the reference information provided by the central decision maker. For a quadratic optimization objective, we provide the optimal control using dynamic programming and show that the optimal controller can be separated into a noncausal feedforward term (generated by the central decision maker) plus a feedback term (generated by the on-site controller) which has causal access to the controls applied without any loss of performance. We show that the feedforward control is the solution of a deterministic quadratic program, i.e., certainty equivalence holds. We later study the problem of transmission of the feedforward controls to the on-site controller over a discrete noiseless channel. We formulate and solve an optimization problem for the optimal time-varying and time invariant uniform quantization of the feedforward control signals sent by the central decision maker to the on-site controller over a communication network.**

I. Introduction

A. Problem Definition

We consider a discrete-time linear system with the following dynamics.

$$x_{P,t+1} = A_P x_{P,t} + B_P u_t + w_t, \quad z_{P,t} = C_P x_{P,t},$$

where (A, B) is controllable and (A, C) is observable, w_t is an i.i.d. Gaussian noise process, with covariance matrix N_w and $z_{P,t}$ is scalar. The path to be followed is given by

$$\{z_{R,t}, \quad 0 \leq t \leq T\}.$$

The tracking error z is equal to:

$$z_t = z_{P,t} - z_{R,t}.$$

The control has the observation

$$y_{C,t} = z_{P,t} + v_t,$$

where v_t is a zero-mean Gaussian observation noise with covariance N_v. The controller function, $\mu(.)$ has access to the information vector I_t,

$$u_t = \mu(I_t),$$

where I_t consists of the initial state, reference trajectory, received observations, and the past controls (Fig. 1).

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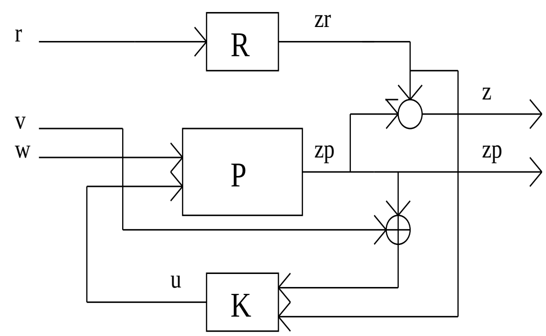

Fig. 1: Basic system structure[1] .

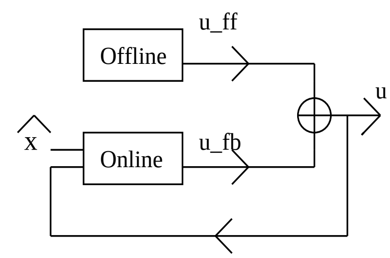

Fig. 2: Separation of the control as feedforward and feedback terms, $\hat{x}$ denotes the estimation output.

$$I_t = \{x_0; z_{R,0}, z_{R,1}, \ldots, z_{R,T}; z_{P,0} + v_0, z_{P,1} + v_1, \ldots, z_{P,t} + v_t; u_0, u_1, \ldots, u_{t-1}\} \quad (1)$$

Given the restrictions on the information vector for all the control terms, we will be considering the minimization of the following cost measure.

$$J_0 = \min_{u_0^T := u_0, u_1, \ldots, u_T} \sum_{t=0}^{T} E[(z_t)^2 + u_t^T R u_t]. \quad (2)$$

We will express the optimal control as a superposition of feedforward and feedback terms (Fig. 2), where feedforward term is defined as the conditional expectation $u_{ff,t} := E[u_t|I_0]$.

Later in the paper we will study the optimal quantization of feedforward control terms over a channel(Fig. 3) such as the widely used CAN (Controller Area Network). Let z_q be the additional deterministic error due to the quantization of the feedforward control signals. We seek the solution to the following optimization problem:

$$\min_{\mathcal{Q}} E[||z_q||_2^2],$$

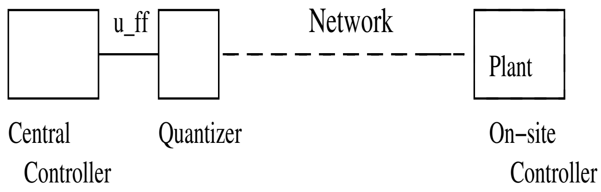

Fig. 3: Transmission of the feedforward control over the network.

subject to

$$\sum_{i=1}^{T} R_i \leq K.$$

where $\mathcal{Q}$ is the set of uniform scalar quantizers, $||.||_2$ denotes the Euclidean (l_2) norm, R_i is the rate per control signal and K is the total rate constraint imposed by the network. Throughout, fixed-rate encoding is used and thus, $R_i = \log_2(K_i)$, where K_i is the number of quantization levels.

B. Relevant Literature

There is an extensive literature on tracking and feedforward control design problems. For a general control objective, *feedforward* control is an effective design scheme if the disturbances are measurable in advance, whereas *feedback* control acts on measured outputs of a system [2], [3]. Such design tools can also be applied in the tracking problem. The performance of a tracking controller depends on the constraints and the information structure under which the controls are generated; such as the availability of causal and non-causal information on the reference path and disturbances, the assumptions on system and observation noises, and communication constraints in case the control is remote. Some of the relevant studies on tracking control design are the following. References [2], [3] study the necessary and sufficient conditions on the existence of feedforward control for asymptotic tracking, stability and invertibility. Reference [7] has studied the H_∞ optimal controller for a nonlinear plant linearized around the operating points fed by a deterministic control. Reference [6] studies the problem of H_∞ tracking with both causal and noncausal reference signals being available and it is shown that the best causal controller is as good as the best noncausal H_∞ controller if the system is minimum phase. Reference [10] provides a polynomial matrix solution to the optimal feedback and feedforward control when a measurable disturbance acts on the system. Reference [8] studies the design of the optimal feedforward tracking controller design using a frequency domain approach.

Our paper is different from the above in that, the reference is available noncausally to the controller, and the system is noisy. Furthermore, the reference is not necessarily the output of a linear plant, for instance it can be the output of a nonlinear computerized scheduler/planner. We provide a finite horizon time-domain approach using dynamic programming. We further provide a new design scheme for networked control systems, based on the feedforward and feedback separation of the control. Such a separation is required, since in practical control systems, the remote controller can be assumed to be computationally incapable of performing complex tasks. In the proposed design, the *feedback* term is generated by the remote controller which has noisy observations from the plant, and the *feedforward* term is provided by a central decision maker which has the noncausally available trajectory information. We first provide the solution to the optimal controller and then separate the optimal solution. The feedforward control needs to be transmitted to the plant, however; an encoding problem for which is also posed and solved.

Networked control systems have been a major research area in the past few years. In this paper we also provide an optimization problem for the encoding of the feedforward control signals. In the encoding of the controls, however, we will restrict our analysis to scalar quantizers, since the computational capability of the plant (the decoder in this case) is assumed to be limited.

We start off with the the optimal control evaluation and feedback-feedforward separation through dynamic programming in section II. We then study the encoding problem in section III, followed by simulations in section IV and end with the concluding remarks in section V.

II. Optimal Control and Feedback-Feedforward Separation

In this section we provide the solution to the optimal control problem and show that it is separable into two components without any loss of optimality. We first provide a definition:

Definition 2.1: Suppose there is no noise in the system, neither in the process nor in the observations and that the optimal control is given by

$$u_k = \Phi_k(x_k).$$

Certainty Equivalence holds if the closed loop optimal control has the same form as the deterministic optimal control with the state being replaced by its estimate.

$$u_k^{CE} = \Phi_k(\hat{x}_k)$$

We also note that certainty equivalence implies the weaker notion of *separation of estimation and control*. We now state the main results of this section.

Theorem 2.1: For the minimization of the quadratic cost problem (2) introduced above, the optimal control for any time $0 \leq t \leq T-1$ is given by

$$u_t = -l_t^{-1}(m_t E[x_{P,t}|I_t] - h_t(z_{R,0}^T)),$$

where

$$l_t = (R + \sum_{\tau=t+1}^{T} B^T A^{\tau-t-1T} m_\tau^T K_\tau m_\tau A^{\tau-t-1} B + \sum_{\tau=t+1}^{T} (C_P A_P^{\tau-t-1} B_P)^T (C_P A_P^{\tau-t-1} B_P)) \quad (3)$$

$$m_t = \sum_{\tau=t+1}^{T} B^T A^{\tau-t-1T} m_\tau^T K_\tau m_\tau A^{\tau-t} + \sum_{\tau=t+1}^{T} (C_P A_P^{\tau-t-1} B_P)^T (C_P A_P^{\tau-t}), \quad (4)$$

$$h_t = \sum_{\tau=t+1}^{T} B^T A^{\tau-t-1T} m_\tau^T K_\tau h_\tau + \sum_{\tau=t+1}^{T-1} (C_P A_P^{\tau-t-1} B_P)^T z_{R,\tau}, \quad (5)$$

and

$$K_t = -l_t^{-T}, \quad (6)$$

with

$$K_T = 0.$$

Remark: As is observed in (5), there is a strict dependency on the future trajectory values, since $C_P A_P^k B_P$ cannot be zero $\forall k \geq 1$. This follows from the assumptions on the plant being observable and controllable [9]. Thus, non-causal access to the reference path improves the performance. ⋄

Theorem 2.2: The optimal control is separable into a feedforward control and a feedback control with causal access to the previously applied controls without any loss of optimality. The feedback control is given by

$$u_{fb,t} = \eta_t [E[x_{P,t}|I_t] - E[x_{P,t}|I_0]] \quad (7)$$

where

$$\eta_t = -l_t^{-1} m_t.$$

The optimal feedforward control is

$$u_{ff,t} = E[u_t|I_0]$$

Furthermore, the optimal feedforward control, is the optimal deterministic l_2 minimizing controller (i.e., certainty equivalence control). The optimal feedforward controller will be of one degree of freedom and only depend on the reference path values.

We devote the rest of this section to the proofs of the theorems.

A. Proof of Theorem 2.1

We define

$$J_\tau(I_\tau) = \min_{u_\tau} \{ \sum_{t=\tau}^{T} E[(z_t)^2 + u^T R u + J_{\tau+1} | I(\tau)] \}.$$

By dynamic programming at time $T-1$ we will have

$$J_{T-1}(I_{T-1}) = \min_{u_{T-1}} E[(C_P x_{P,T-1} - z_{R,T-1})^2 + u_{T-1}^T R u_{T-1} + (C_P A_P x_{P,T-1} + C_P B_P u_{T-1} + C_P w_{T-1} - z_{R,T})^2 | I_{T-1}, u_{T-1}]$$

The first term on the right hand side is not a function of the controller. Therefore the controller minimizes

$$E[u_{T-1}^T R u_{T-1} + (C_P A_P x_{P,T-1} + C_P B_P u_{T-1} + C_P w_{T-1} - z_{R,T})^2 | I_{T-1}, u_{T-1}]. \quad (8)$$

Taking the gradient with respect to u_{T-1}, we obtain

$$E[2Ru_{T-1} + 2(C_P B_P)^T (C_P B_P u_{T-1} + C_P A_P x_{P,T-1} + C_P w_{T-1} - z_{R,T}) | I_{T-1}]$$

Thus, we compute the optimal control as:

$$u_{T-1} = -(R + (C_P B_P)^T C_P B_P)^{-1} (C_P B_P)^T . E[C_P A_P x_{P,T-1} + C_P w_{T-1} - z_{R,T} | I_{T-1}]$$

The expectation depends on the information vector available to the controller at time $T-1$. We will discuss the estimation problem later.

Given r_0^T, then $z_{R,T}$ is deterministic (for the reference path is noncausally available); therefore, one can take it out of the expectation:

$$u_{T-1} = (R + (C_P B_P)^T C_P B_P)^{-1} (C_P B_P)^T z_{R,T} - (R + (C_P B_P)^T C_P B_P)^{-1} (C_P B_P)^T . E[C_P A_P x_{P,T-1} + C_P w_{T-1} | I_{T-1}]$$

Since w_{T-1} is independent of I_{T-1}, we have u_{T-1} as

$$(R + (C_P B_P)^T C_P B_P)^{-1} (C_P B_P)^T (z_{R,T}) - (R + (C_P B_P)^T C_P B_P)^{-1} (C_P B_P)^T C_P A_P E[x_{P,T-1} | I_{T-1}]$$

or

$$u_{T-1} = -(l_{T-1})^{-1} (m_{T-1} E[x_{P,T-1} | I_{T-1}] - h_{T-1}),$$

where

$$l_{T-1} = (R + (C_P B_P)^T C_P B_P),$$

$$m_{T-1} = (C_P B_P)^T C_P A_P,$$

and

$$h_{T-1} = (C_P B_P)^T z_{R,T}$$

Note that the optimal policy is an affine function of the conditional expectation. We have

$$J_{T-1}(I_{T-1}) = (C_P x_{P,T-1} - z_{R,T-1})^2 + ((C_P)' C_P E[w_{T-1} w'_{T-1}]) + (C_P A_P x_{P,T-1} - z_{R,T})^2 + E[(h_{T-1}(z_{R,0}^T) - m_{T-1} x_{P,T-1})^T K_{T-1} . (h_{T-1}(z_{R,0}^T) - m_{T-1} x_{P,T-1})] + E[(x_{P,T-1} - E[x_{P,T-1} | I_{T-1}])^T P_{T-1} . (x_{P,T-1} - E[x_{P,T-1} | I_{T-1}]) | I_{T-1}],$$

where

$$K_{T-1} = -l_{T-1}^{-1},$$

and

$$P_{T-1} = m_{T-1} l_{T-1}^{-1} m_{T-1}.$$

Thus the cost at time $T-1$ can be expressed as the cost due to the estimation and the control cost. Now the DP equation for time $T-2$ is

$$J_{T-2}(I_{T-2}) = \min_{u_{T-2}} E[(C_P x_{P,T-2} - C_R x_{R,T-2})^2 + u_{T-2}^T R u_{T-2} + J_{T-1}(I_{T-1})]$$

A control does not have a dual effect ([1], [5]) if the state estimation error process is independent of the applied control. In our case, we note that there is no dual effect, for the estimation error at time $T-1$ is not a function of the control applied at time $T-2$. The controller has access to the control received by the plant, and since that control is additive, the state estimation error is not affected by the control policy and is merely a function of the disturbance processes. Expressing J_{T-1} as a function of u_{T-2}, we obtain:

$$\begin{aligned} J_{T-2}(I_{T-2}) = \min_{u_{T-2}} \{&(C_P x_{P,T-2} - C_R x_{R,T-2})^2 \\ &+ u_{T-2}^T R u_{T-2} + (C_P x_{P,T-1} - z_{R,T-1})^2 \\ &+ (C_P A_P x_{P,T-1} - z_{R,T})^2 + E[w_{T-1}^T C_P^T C_P w'_{T-1}] \\ &+ E[(h_{T-1}(z_{R,0}^T) - m_{T-1} x_{P,T-1})^T \\ &. K_{T-1}(h_{T-1}(z_{R,0}^T) - m_{T-1} x_{P,T-1})] \\ &+ E[(x_{P,T-1} - E[x_{P,T-1}|I_{T-1}])^T P_{T-1} \\ &. (x_{P,T-1} - E[x_{P,T-1}|I_{T-1}])|I_{T-1}], \end{aligned}$$

One can exclude the last term (the state estimation error component) from the minimization with respect to u_{T-2}, as there is no dual effect of the control, i.e. $x_t - E\{x_t|I_t\}$ is not a function of the control signal. Thus u_{T-2} minimizes:

$$\begin{aligned} E[&u_{T-2}^T R u_{T-2} + (C_P x_{P,T-1} - z_{R,T-1})^2 \\ &+ (C_P A_P x_{P,T-1} - z_{R,T})^2 + (h_{T-1}(z_{R,0}^T) \\ &- m_{T-1} x_{P,T-1})^T K_{T-1}(h_{T-1}(z_{R,0}^T) \\ &- m_{T-1} x_{P,T-1})|I_{T-2}] \end{aligned}$$

Using $x_{P,T-1} = A_P x_{P,T-2} + B_P u_{T-2} + w_{T-2}$, we obtain:

$$\begin{aligned} &E[u_{T-2}^T R u_{T-2} \\ &+ (C_P[A_P x_{P,T-2} + B_P u_{T-2} + w_{T-2}] - z_{R,T-1})^2 \\ &+ (C_P[A_P^2 x_{P,T-2} + A_P B_P u_{T-2} + A_P w_{T-2}] \\ &- z_{R,T})^2 + (h_{T-1}(z_{R,0}^T) - m_{T-1}[A_P x_{P,T-2} \\ &+ B_P u_{T-2} + w_{T-2}])^T K_{T-1}(h_{T-1}(z_{R,0}^T) \\ &- m_{T-1}[A_P x_{P,T-2} + B_P u_{T-2} + w_{T-2}])|I_{T-2}] \end{aligned}$$

Again taking the gradient and setting it to zero we will obtain the optimal control.

$$u_{T-2} = -l_{T-2}^{-1}(E[(m_{T-2} x_{P,T-2} - h_{T-2}(z_{R,0}^T))|I_{T-2}], \quad (9)$$

where

$$\begin{aligned} l_{T-2} &= (R + B_P^T(C_P^T C_P + (C_P A_P)^T C_P A_P) B_P \\ &\quad + (m_{T-1} B_P)^T K_{T-1}(m_{T-1} B_P)) \\ m_{T-2} &= (C_P B_P)^T C_P A_P + B_P^T (C_P A_P)^T C_P A_P A_P \\ &\quad + (m_{T-1} B_P)^T K_{T-1} A_P \\ h_{T-2} &= (C_P B_P)^T z_{R,T-1} + (C_P A_P B_P)^T z_{R,T} \\ &\quad + (m_{T-1} B)' K_{T-1} h_{T-1} \end{aligned}$$

Defining

$$N_{T-2} = l_{T-2}^{-1},$$

we can write the expectation as the difference between the state and the estimation error. We thus obtain:

$$\begin{aligned} &[m_{T-2} x_{P,T-2} - h_{T-2}(z_{R,0}^T)]^T K_{T-2} \\ &.[m_{T-2} x_{P,T-2} - h_{T-2}(z_{R,0}^T)] \\ &+ (x_{P,T-2} - E[x_{P,T-2}])^T P_{T-2}(x_{P,T-2} \\ &- E[x_{P,T-2}]) + (C_P(A_P x_{P,T-2}) - C_R x_{R,T-1})^2 \\ &+ (h_{T-1}(z_{R,0}^T) - C_P A_P(A_P x_{P,T-2}))^T K_{T-1} \\ &.(h_{T-1}(z_{R,0}^T) - C_P A_P(A_P x_{P,T-2}))|I_{T-2}] \\ &+ E[w_{T-2}^T[(m_{T-1})^T K_{T-1}(m_{T-1}) + C_P^T C_P] w_{T-2}] \\ &+ E[w_{T-1}^T C_P^T C_P w_{T-1}] + E[(x_{P,T-1} \\ &- E[x_{P,T-1}|I_{T-1}])^T \\ &. P_{T-1}(x_{P,T-1} - E[x_{P,T-1}|I_{T-1}])|I_{T-1}], \end{aligned}$$

with,

$$K_{T-2} = -N_{T-2}$$

$$P_{T-2} = m_{T-2}^T N_{T-2} m_{T-2}.$$

Thus, one can obtain recursions for any time t.

$$\begin{aligned} J_t = &[m_t x_{P,t} - h_t(z_{R,0}^T)]^T K_t .[m_t x_{P,t} - h_t(z_{R,0}^T)) \\ &+ (x_{P,t} - E[x_{P,t}])^T P_t (x_{P,t} - E[x_{P,t}]) \\ &+ \sum_{\tau=t+1}^{T} [m_\tau A_P^{\tau-t} x_{P,t} - h_\tau(z_{R,0}^T)]^T K_\tau \\ &.[m_\tau A_P^{\tau-t} x_{P,t} - h_\tau(z_{R,0}^T)] \\ &+ \sum_{\tau=t}^{T} (z_{R,\tau} - C_P A_P^{\tau-t} x_{P,t})^T (z_{R,\tau} - C_P A_P^{\tau-t} x_{P,t}) \\ &\sum_{\tau=t}^{T} E[w_\tau^T((m_\tau A_P^{\tau-t})^T K_\tau m_\tau A_P^{\tau-t} \\ &+ [C_P A_P^{\tau-t}]^T [C_P A_P^{\tau-t}] w_\tau] \\ &+ \sum_{\tau=t+1}^{T} E[(x_{P,\tau} - E[x_{P,\tau}|I_\tau])^T P_\tau \\ &.(x_{P,\tau} - E[x_{P,\tau}|I_\tau])|I_\tau]. \end{aligned} \quad (10)$$

The recursions for the control terms follow. ◇

Remark The expected cost imports terms from the future. We also note that, the estimation error strictly increases the cost of the system, in an additive way. ◇

B. Proof of Theorem 2.2

Rewriting the optimal control u_t as

$$u_t = \alpha_t \beta_t E[x_{P,t}|I_t] + \alpha_t h_t(z_{R,0}^T)) \quad (11)$$

where

$$\alpha_t = l_t^{-1}$$

and

$$\beta_t = -m_t,$$

we can write the control as

$$u_t = (E[u_t|I_0]) + (u_t - E[u_t|I_0])$$

which will lead to the feedforward and the feedback terms as follows:

$$u_{ff,t} = E[u_t|I_0]$$

and

$$u_{fb,t} = u_t - E[u_t|I_0].$$

In this case, we will have

$$E[u_t|I_0] = \alpha_t \beta_t E[E[x_{P,t}|I_t]|I_0] \\ +\alpha_t h_t(z_{R,0}^T))] \quad (12)$$

To compute the expectation we have, we provide a recursion for the state evolution:

$$x_{P,t+1} = A_P x_{P,t} + B_P(\alpha_t \beta_t E[x_{P,t}] + \alpha_t h_t(z_{R,0}^T))) + w_t,$$

which can be written as:

$$x_{P,t+1} = (A_P + B_P \alpha_t \beta_t) x_{P,t} - B_P(\alpha_t \beta_t) \\ (x_{P,t} - E[x_{P,t}]) + B_P \alpha_t h_t(z_{R,0}^T))) + w_t,$$

Thus, the state at some time k can be expressed as:

$$x_{P,k+1} = (\prod_{t=0}^{k}(A_P + B_P \alpha_t \beta_t) x_{P,0} \\ -\sum_{t=0}^{k}\{\prod_{t=l}^{k} B_P(A_P + B_P\alpha_t\beta_t)\}(\alpha_l \beta_l)(x_{P,l} - E[x_{P,l}]) \\ +\sum_{t=0}^{k}\{\prod_{t=l}^{k}(A_P + B_P\alpha_t\beta_t)\}B_P\alpha_l h_l(z_{R,0}^T))) \\ +\sum_{t=0}^{k}\{\prod_{t=l}^{k}(A_P + B_P\alpha_t\beta_t)\}w_l.$$

Here $x_{P,t} - E[x_{P,t}|I_t]$ and w_t are zero-mean processes. The estimation errors are orthogonal to the estimates. Therefore the expectation of the state at time 0 will be

$$E[x_{P,k+1}|I_0] = (\prod_{t=0}^{k}(A_P + B_P \alpha_t \beta_t) x_{P,0} \\ +\sum_{t=0}^{k}\{\prod_{t=l}^{k}(A_P + B_P\alpha_t\beta_t)\}B_P\alpha_l h_l(z_{R,0}^T))) \quad (13)$$

Thus, the noise terms are taken out, and the feedforward control is a certainty equivalence control.

The online feedback optimal control at a given time t is:

$$u_{fb,t} = u_t - (E[u_t|I_0])$$

which becomes

$$u_{fb,t} = \alpha_t \beta_t E[x_{P,t}|I_t] + \alpha_t h_t(z_{R,0}^T))] \\ -\alpha_t \beta_t (E[u_t|I_0]). \quad (14)$$

And finally we obtain

$$u_{fb,t} = \alpha_t \beta_t [E[x_{P,t}|I_t] + \alpha_t h_t(z_{R,0}^T))] \\ -\alpha_t \beta_t (E[x_{P,t}|I_0]) - E[\alpha_t h_t(z_{R,0}^T)|I_0]. \quad (15)$$

The deterministic parts cancel each other leading to

$$u_{fb,t} = \alpha_t \beta_t [E[x_{P,t}|I_t] - E[x_{P,t}|I_0]] \quad (16)$$

Therefore the optimal controller can be implemented as two separate components; a feedforward term and a feedback term (However we should note that the estimator in the feedback term should have *causal* access to the last applied control). The optimal feedforward term is now precomputed.

Suppose there is no noise in the system evolution and no state uncertainty. Then the dynamic programming recursion would cross out the noise and the state estimation error terms and the same recursion would be obtained for the control terms. Note that this is the same as arguing that *certainty equivalence* holds in this case. Another interpretation for this argument is the following. Let $u_0^j := \{u_0, u_1, \ldots, u_j\}$. The feedforward seeks to minimize

$$\min_{u_0^T} E[J_0|I_0] = \min_{u_0^T}\{\ldots\{\{z_T^2|(I_0, u_0^{T-1})\} \\ +z_{T-1}^2 + u_{T-1}^T R u_{T-1}|(I_{T-2}, u_0^{T-2})\}\ldots \\ +z_0^2 + u_0^T R u_0|(u_0, I_0)\} \quad (17)$$

Telescoping outwards the conditionings for instance for $T-1$, we obtain

$$E[(C_P x_{P,T-1} - C_R x_{R,T-1})^2 + u_{T-1}^T R u_{T-1} \\ (C_P A_P x_{P,T-1} + C_P B_P u_{T-1} + C_P w_{T-1} \\ -z_{R,T})^2 | I_0, u_0^{T-2}], \quad (18)$$

the minimization of which yields to the same control obtained in (9). ◇

Remark Note that α_t and β_t are independent of the reference path values; they are solely functions of plant dynamics. ◇

Thus for the optimal controller, we can provide an off-line deterministic l_2 minimizing controller and supplement this with a noise rejecting filter as the optimal solution.

C. On Estimation

For the sake of completeness we briefly study the estimation problem. As was studied above, the feedback control law is independent of the estimation, due to the separation principle. For the optimal estimation, for a linear system, one can use Kalman filtering if the noise statistics are Gaussian. The optimal feedback control estimator needs causal information on the applied controls to perform the estimation. In the simulation studied in this paper we use time-varying Kalman filter for the estimation.

III. Optimal quantization of the feedforward signals

We consider a plant that is controlled remotely over a finite capacity network. In such a system the plant might not be able to make use of the noncausal information available to it. However, by separating the control to two components, the central decision maker sends the feedforward commands to the plant, and the plant can merely be in charge of suppressing the noise and following the feedforward commands generated by the decision maker.

The central decision maker cannot send the feedforward controls with arbitrary precision over the network, hence we need to encode the controls. While implementing this, the decoding should be simple as well. Therefore, we restrict the encoding to be within the class of uniform scalar quantizers, as this is the computationally simplest quantizer.

Let $z^q_{R,t}$ be the plant position with quantized feedforward control. Define z_q as the additional error due to the quantization of the feedforward signals, i.e.

$$z_{q,t} = z_{R,t} - z^q_{P,t}.$$

We seek the solution to,

$$\min E[||z_q||_2^2], \tag{19}$$

subject to

$$\sum_{i=1}^{T} R_i \leq K$$

where R_i is the rate (in the sense of the logarithm of the number of levels; fixed length encoding) per control signal and K is the total rate constraint imposed by the network.

We have the following theorem summarizing the results of this section. In the theorem we assume that the quantization errors are independent of the quantized values, which is a common assumption in high-rate quantization [11].

Theorem 3.1: Let A_{cnv} be the convolution matrix given by:

$$A_{cnv}(i,j) = CA^{i-j}B1_{(i \geq j)},$$

where $1_{(.)}$ is the indicator function. And let

$$a_j = \sum_i |A_{cnv}(i,j)|^2 \quad 1 \leq j \leq T,$$

and

$$M = \inf\{L : |u_i| \leq L/2 \quad \forall i\}$$

Then, the optimal solution to (19) among the class of time-invariant scalar uniform quantizers has the additional cost of

$$D_{inv}(K) = [\sum_i (a_i)^2] M^2 2^{-2K/T}.$$

The optimal time varying uniform scalar quantizer will have the additional cost as

$$D_{tv}(K) = \sum_i M^2 2^{2/T(\sum \log_2(a_i))} 2^{-2K/T}.$$

Proof:

We have $\hat{u} = u + u_e$, where $\hat{u}$ is the quantized control and u_e is the quantization error. The derailment in the path due to the quantization error is equal to $z_q = A_{cnv}u_e$. We seek to minimize

$$E[||A_{cnv}u_e||_2]$$

which is equal to

$$E[\sum_i [\sum_j A_{cnv}(i,j) u_e(j)]^2]$$

Using the independence of the quantization errors, the term above is equivalent to

$$E[\sum_j [\sum_i A_{cnv}(i,j)^2] u_e(j)^2]$$

Using a uniform quantizer for each of the controls, we obtain,

$$\min_{l_0,l_1,\ldots,l_T} \sum_j |M|/2^{2l_j} \sum_i |A_{cnv}(i,j)|^2$$

subject to

$$\sum \log_2(l_j) \leq K$$

Defining $x_i := \log_2(l_i)$, we have the problem of

$$\min \sum (a_i)^2 M^2 2^{-2x_i}$$

subject to

$$\sum x_i \leq K.$$

This problem is convex in x_i, constraint qualifications hold and the unique optimal solution can be obtained using a Lagrangian approach. Define

$$L(x,\lambda) := \min \sum a_i^2 M^2 2^{-2x_i} + \lambda \sum x_i,$$

where $\lambda \geq 0$. If we take the derivative with respect to x_i, we obtain the following necessity condition:

$$-2(a_i)^2 M^2 ln(2) 2^{-2x_i} + \lambda = 0$$

which suggests that $(a_i)/2^{x_i}$ should be constant for all time stages i and that $\lambda > 0$. We then have

$$x_i = \log_2(a_i) - \log_2(\alpha),$$

for some α which satisfies the constraint equation:

$$\sum x_i = K,$$

due to the complementary slackness condition. We thus have

$$\alpha = 2^{1/T[\sum \log_2(a_i) - K]}$$

If we had restricted the quantizers to be strictly time invariant then for each of the controls we would have used $R_i = K/T$ bits, and l_i would be fixed for all time stages. In this case all the controls would have the same expected quantization error and the cost would be

$$\sum (a_i)^2 M^2 2^{-2K/T}.$$

This concludes the proof. ◇

Remark Note that there is an exponential decrement in the derailment due to the quantization error in both cases. However, since the first control terms are more impactful than the latter ones (as can also be seen from the fact that the convolution matrix is lower-triangular) using a time varying scheme improves the performance. ◇

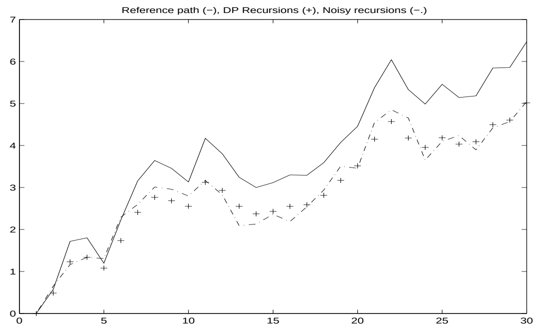

Fig. 4: Reference, noisy and noiseless paths, R=0.05

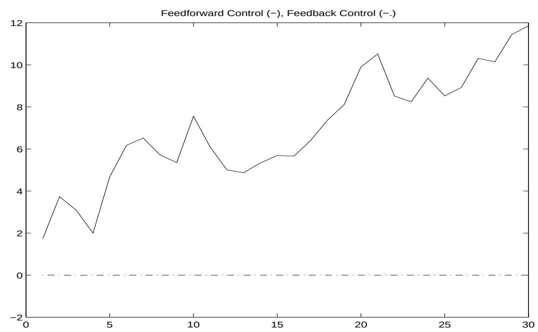

Fig. 5: Feedforward and feedback controls, R=0.05

IV. Simulations

A three-dimensional plant with the following matrices is used in the simulations.

$$A = \begin{bmatrix} 0.4 & 0.01 & -0.05 \\ 0.02 & 0.5 & 0.01 \\ -0.02 & 0.1 & 0.3 \end{bmatrix}$$

$$B = 1/2 \begin{bmatrix} 1 & 0.1 & 0.1 \end{bmatrix}^T$$

$$C = 1/2 \begin{bmatrix} 1 & 0.2 & 1 \end{bmatrix}.$$

The penalty term for the control, R, is adjusted to provide the two different sets of plots; Figs. 4, 5 and 6, 7.

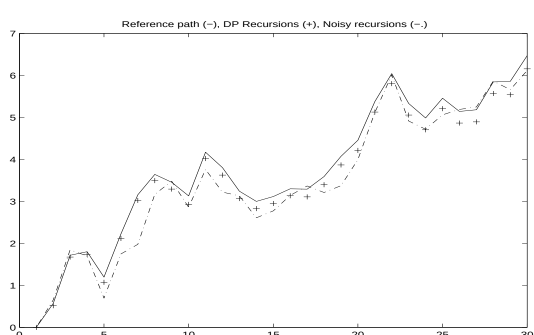

Fig. 6: Reference, noisy and noiseless paths, R=0.000005

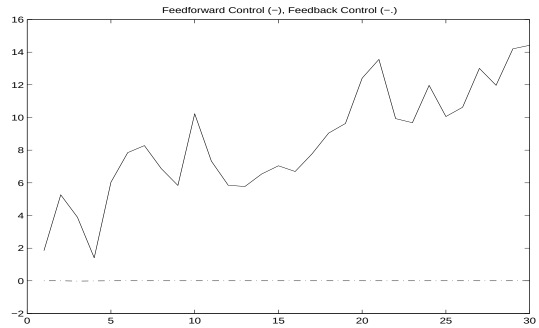

Fig. 7: Feedforward and feedback controls, R=0.000005

V. Conclusion

In this paper we constructed the optimal l_2 feedback and feedforward controls, and showed that there is no loss of optimality as a result of the optimal separation of the controller to a noncausal and complex feedforward and a standard LQR feedback component. We then computed the optimal time varying and time invariant uniform quantizers minimizing the expected distortion under a data rate constraint.

All in all, we designed a tracking controller in a networked system setup. We have had three components in the cost for tracking; the deterministic error, the stochastic estimation errors due to the noise components in the system and an additional quantization error due to the communication network constraints. All the effects of these are quantifiable and appear additively in the outcome.

VI. Acknowledgements

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