Networked Control Systems with Unbounded Noise under Information Constraints

by

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Abstract

We investigate the stabilization of unstable multidimensional partially observed single-station, multi-sensor (single-controller) and multi-controller (single-sensor) linear systems controlled over discrete noiseless channels under fixed-rate information constraints. Stability is achieved under communication requirements that are asymptotically tight in the limit of large sampling periods. Through the use of similarity transforms, sampling and random-time drift conditions we obtain a coding and control policy leading to the existence of a unique invariant distribution and finite second moment for the sampled state. We use a vector stabilization scheme in which all modes of the linear system visit a compact set together infinitely often.

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Chapter 1

Introduction

1.1 Networked Control Systems As Dynamic Teams

Networked control systems have been extensively studied and we refer the reader to [1] for an overview of the theory.

Networked control systems can be viewed as stochastic dynamic team problems. The mathematical framework for such problems typically involves four components. They are

1. decision makers;
2. uncertainty;
3. information structure;
4. cost function.

For our purposes, decision makers are either sensors or controllers. Sensors receive measurements on the environment and decide what to send to one or more controllers. The controllers receive messages from one or more sensors and apply an action which influences the environment. Typically, we introduce a state variable which the controllers attempt to influence.

There is an element of uncertainty in the measurements taken by the sensors, in the value taken by the state, or in both. In this report, uncertainty will be introduced by additive plant and observation noise, taking values according to some probability distribution.

The information structure defines what information is available to the sensors and controllers at each time stage and thus determines how they can interact and what policies they can adopt.

Unlike in adversarial game theory, in team problems the decision makers work together towards the goal of minimizing some cost function or achieving some performance criterion. In this report, the cost function is the rate of a discrete noiseless channel between the sensors and the controllers. The goal is to minimize the rate while maintaining some form of stochastic stability for the system. We will make a precise mathematical definition of several forms of stochastic stability later on.

By definition, a team problem is dynamic [2] if the information available to a decision maker is affected by the actions of some decision maker. We are interested in a class of sequential dynamic team problems. In a sequential scheme, the order in which agents act is deterministic. For a problem with M stations that are both sensors and controllers, the system equations are given by

$$\begin{aligned}\mathbf{x}_{t+1} &= f_t(\mathbf{x}_t, \mathbf{w}_t, \mathbf{u}_t^1, \ldots, \mathbf{u}_t^M)\\ \mathbf{y}_t^j &= g_t^j(\mathbf{x}_t, \mathbf{v}_t^j, \mathbf{u}_{t-1}^1, \ldots, \mathbf{u}_{t-1}^M)\end{aligned}$$

for some set of functions $\{f_t\}$ and $\{g_t^j\}$ with $f_t : \mathbb{X}_t^2 \times \mathbb{U}_t^M \to \mathbb{Y}_t$ and $g_t : \mathbb{X}_t \times \mathbb{Y}_t^j \times \mathbb{U}_{t-1}^M \to \mathbb{Y}_t^j$ for $1 \leq j \leq M$ and $t \in \mathbb{N}$. The state variable is $\mathbf{x}_t \in \mathbb{X}_t$, the control action of station j at time t is $\mathbf{u}_t^j \in \mathbb{U}_t$ and its measurement or observation is $\mathbf{y}_t^j \in \mathbb{Y}_t^j$. The processes $\{\mathbf{w}_t\}$ and $\{\mathbf{v}_t^j\}$ are sequences of noise variables which take values according to some probability distribution. The initial state $\mathbf{x}_0$ is also determined by a probability distribution. Thus, there is an element of randomness in the values taken by the state and the measurements made by the stations, making the above a stochastic problem.

Example. Consider the single-station, fully observed discrete LTI system described by the equations

$$\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{B}\mathbf{u}_t + \mathbf{w}_t, \qquad \mathbf{y}_t = \mathbf{x}_t + \mathbf{v}_t, \tag{1.1}$$

where $\mathbf{x}_t \in \mathbb{R}^n$, $\mathbf{u}_t \in \mathbb{R}^m$ and $\mathbf{y}_t \in \mathbb{R}^n$ are the state, control action and sensor observation at time t respectively. The matrices $\mathbf{A}$ and $\mathbf{B}$ are of appropriate dimension. The processes $\{\mathbf{w}_t\}$ and $\{\mathbf{v}_t\}$ are each independent identically distributed (i.i.d.) sequences of random vectors. At time t both $\mathbf{w}_t$ and $\mathbf{v}_t$ are independent of the state $\mathbf{x}_t$ and each other.

At time t, we allow the sensor to send an encoded value $q_t \in \{1, \ldots, N_t\}$ for some $N_t \in \mathbb{N}$ to the controller over a discrete noiseless channel. We define the rate at time t to be the number of bits needed to send the message. More precisely, the rate at time t is $R_t = \log_2(N_t)$. The average rate is given by

$$R_{\text{avg}}(N) = \frac{1}{N} \sum_{t=0}^{N-1} R_t.$$

Information structure. For a process $\{\mathbf{x}_t\}$ we define $\mathbf{x}_{[a,b]} = \{\mathbf{x}_a, \mathbf{x}_{a+1}, \ldots, \mathbf{x}_b\}$. At time t, the sensor maps its information $I_t^s := \mathbf{y}_{[0,t]} \to q_t$ and the controller maps its information $I_t^c := q_{[0,t]} \to \mathbf{u}_t$.

The task of the sensor is to map the continuous observations $\mathbf{y}_{[0,t]}$ to the discrete encoded value q_t. The controller must then form a continous estimate $\hat{\mathbf{x}}_t$ based on this value. This is accomplished through the application of a quantizer, which we define presently.

Definition 1.1.1. *A k-dimensional N-bin vector quantizer is a mapping $Q : \mathbb{R}^k \to \mathcal{C}$ where $\mathcal{C} = \{c_1, \ldots, c_N\} \subset \mathbb{R}^k$ is called the codebook. The quantizer is characterized by the sequence of bins $\{\mathcal{B}_i\}_{i=1}^N$ where $\mathcal{B}_i = \{\mathbf{x} \in \mathbb{R}^k \mid Q(\mathbf{x}) = c_i\}$. The bins form a partition of $\mathbb{R}^k$. That is, $\cup_{i=1}^N \mathcal{B}_i = \mathbb{R}^k$ and $\mathcal{B}_i \cap \mathcal{B}_j = \emptyset$ for all $i \neq j$. A scalar quantizer is a vector quantizer of dimension $k = 1$.*

For networked control systems, the design objective is typically stability or optimality. The precise type of stability considered depends on the context of the problem. In this report, we are interested in two strong forms of stability. Given a state process $\{\mathbf{x}_t\}$, satisfying certain conditions, we want to show that

1. $\{\mathbf{x}_t\}$ has a finite second moment and
2. $\{\mathbf{x}_t\}$ is positive Harris recurrent (see Appendix) and has a unique invariant distribution.

The intuition behind the first condition is clear. We want the state to be well behaved in the sense that its expected second moment converges. The second condition is useful because of the ergodic theorem. Let $\{\mathbf{x}_t\}$ be a Markov process, satisfying positive Harris recurrence with invariant measure π. The ergodic theorem, due to Birkhoff, states that almost surely

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} f(\mathbf{x}_t) = \int f(\mathbf{x})\pi(d\mathbf{x}),$$

for all integrable f under π. One reason this theorem is important is because it connects the theory of Markov chains to the theory of stochastic control. In many applications, f is a cost function. In the context of infinite horizon decision problems, the ergodic theorem translates the problem of cost optimization for Markov chains into the problem of stochastic optimization over the set of invariant distributions. We can then take a convex analytic approach (see [3] and [4]).

Before proceeding further, we introduce a fundamental result on stability. References [5], [6] and [7] obtained a lower bound on the average rate of the information transmission for the finiteness of second moments. Their result is generalized in [1] and we reproduce the proof below for convenience.

Theorem 1.1.2. *Suppose in the system* (1.1) *that* $\mathbf{x}_0$ *has finite differential entropy. Then a necessary condition for*

$$\limsup_{t\to\infty} \frac{1}{t} \log(E[\|\mathbf{x}_t\|_2]) \leq 0$$

is that

$$\liminf_{N\to\infty} R_{avg}(N) \geq \sum_{|\lambda_i|>1} \log_2(|\lambda_i|), \tag{1.2}$$

where $\{\lambda_i\}$ *is the set of eigenvalues of* $\mathbf{A}$.

Proof of Theorem 1.1.2: We say that an eigenvalue is stable if it is within the unit circle (modulus strictly less than one) and that it is unstable otherwise. Since the matrix $\mathbf{A}$ can always be diagonalized into two blocks, one with all eigenvalues stable and the other with all eigenvalues unstable, we need only consider the case where $\mathbf{A}$ has all eigenvalues unstable.

We denote the expected norm and covariance matrix of $\mathbf{x}_t$ by

$$\mathbf{S}_t = E[\mathbf{x}_t^T \mathbf{x}_t], \quad \mathbf{\Sigma}_t = E[\mathbf{x}_t \mathbf{x}_t^T].$$

We will use the usual information theory notation. Namely, $H(\cdot)$ denotes discrete entropy, $h(\cdot)$ denotes differential entropy and $I(\cdot;\cdot)$ denotes mutual information. A key fact in the proof is that the entropy of the encoded values $H(q_{[0,T-1]})$ serves as a lower bound on the average rate for information transmission (see Proposition 5.3.1 of [1]). We then have

$$\begin{aligned}
TR_{\text{avg}}(T) &\geq H(q_{[0,T-1]}) = \sum_{t=1}^{T-1} H(q_t | q_{[0,t-1]}) + H(q_0) \\
&\geq \sum_{t=1}^{T-1} \Big(H(q_t | q_{[0,t-1]}) - H(q_t | \mathbf{x}_t, q_{[0,t-1]}) \Big) + H(q_0) \qquad (1.3) \\
&= \sum_{t=1}^{T-1} I(\mathbf{x}_t; q_t | q_{[0,t-1]}) + H(q_0) \\
&= \sum_{t=1}^{T-1} \Big(h(\mathbf{x}_t | q_{[0,t-1]}) - h(\mathbf{x}_t | q_{[0,t]}) \Big) + H(q_0) \\
&= \sum_{t=1}^{T-1} \Big(h(\mathbf{A}\mathbf{x}_{t-1} + \mathbf{B}\mathbf{u}_{t-1} + \mathbf{w}_{t-1} | q_{[0,t-1]}) - h(\mathbf{x}_t | q_{[0,t]}) \Big) + H(q_0) \\
&= \sum_{t=1}^{T-1} \Big(h(\mathbf{A}\mathbf{x}_{t-1} + \mathbf{w}_{t-1} | q_{[0,t-1]}) - h(\mathbf{x}_t | q_{[0,t]}) \Big) + H(q_0) \\
&\geq \sum_{t=1}^{T-1} \Big(h(\mathbf{A}\mathbf{x}_{t-1} + \mathbf{w}_{t-1} | q_{[0,t-1]}, \mathbf{w}_{t-1}) - h(\mathbf{x}_t | q_{[0,t]}) \Big) + H(q_0) \qquad (1.4) \\
&= \sum_{t=1}^{T-1} \Big(h(\mathbf{A}\mathbf{x}_{t-1} | q_{[0,t-1]}, \mathbf{w}_{t-1}) - h(\mathbf{x}_t | q_{[0,t]}) \Big) + H(q_0) \\
&= \sum_{t=1}^{T-1} \Big(h(\mathbf{A}\mathbf{x}_{t-1} | q_{[0,t-1]}) - h(\mathbf{x}_t | q_{[0,t]}) \Big) + H(q_0) \qquad (1.5)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{t=1}^{T-1} \left(\log_2(|\det(\mathbf{A})|) + h(\mathbf{x}_{t-1}|q_{[0,t-1]}) - h(\mathbf{x}_t|q_{[0,t]}) \right) + H(q_0) \\
&= \left(\sum_{t=1}^{T-1} \log_2(|\det(\mathbf{A})|) \right) + h(\mathbf{x}_0|q_0) - h(\mathbf{x}_{T-1}|q_{[0,T-1]}) + H(q_0) \\
&\geq \left(\sum_{t=1}^{T-1} \log_2(|\det(\mathbf{A})|) \right) + h(\mathbf{x}_0|q_0) - h(\mathbf{x}_{T-1}) + H(q_0) \qquad (1.6) \\
&\geq \left(\sum_{t=1}^{T-1} \log_2(|\det(\mathbf{A})|) \right) + h(\mathbf{x}_0|q_0) - \frac{1}{2}\log\left((2\pi e)^n \det(\mathbf{\Sigma}_{T-1}) \right) + H(q_0) \qquad (1.7) \\
&\geq \left(\sum_{t=1}^{T-1} \log_2(|\det(\mathbf{A})|) \right) + h(\mathbf{x}_0|q_0) - \frac{1}{2}\log\left((2\pi e)^n (\frac{1}{n}\mathbf{S}_{T-1})^n \right) + H(q_0). \qquad (1.8)
\end{aligned}
$$

In the above, (1.3) holds because discrete entropy is nonnegative. The inequality (1.4) holds because conditioning cannot increase entropy. Line (1.5) follows since $\mathbf{w}_{t-1}$ is independent of $\mathbf{x}_{t-1}$. The inequality (1.7) holds because, for a given covariance matrix, the Gaussian distribution maximizes entropy. The last step (1.8) is an application of the arithmetic-geometric mean inequality which shows that $\det(\Sigma_t) \leq (\frac{1}{n}Tr(\Sigma_t))^n$. Taking the limit under our assumption that $\lim\sup_{T\to\infty} \frac{1}{T}\log(\mathbf{S}_T) \leq 0$ then gives the result. □

Remark 1.1.3. *It is shown in [7] that* (1.2) *is also a necessary condition for*

$$\sup_{t\in\mathbb{N}} E[\|\mathbf{x}_t\|_2] < \infty.$$

Although we have shown the bound in the fully observed single-station case, it holds for all systems in this report. This follows because in team problems, more information does not hurt performance. Thus, the lower bound in the centralized setting is also a lower bound in the decentralized setting [1]. In light of this fact, we find it convenient to define

$$R_{\min} = \sum_{|\lambda_i|>1} \log_2(|\lambda_i|). \qquad (1.9)$$

1.2 Notation

We denote the indicator function of an event E by 1_E. We will use $\mathbb{R}^{m\times n}$ to denote the space of real $m \times n$ matrices and $\mathbb{R}^n$ to denote the space of real n dimensional vectors. We let $\mathbb{R}^n_+$ be the space of real n dimensional vectors with all entries nonnegative. Unless otherwise stated, all vectors are assumed to be column vectors. For any $\mathbf{x} \in \mathbb{R}^n$ we write $\mathbf{x} = \begin{bmatrix} x^1 & \cdots & x^n \end{bmatrix}^T$ where $x^i \in \mathbb{R}$ is the i^{th} entry. We define the absolute value operation for vectors as the component-wise absolute value. That is, $|\mathbf{x}| = \begin{bmatrix} |x^1| & \cdots & |x^n| \end{bmatrix}^T$. For a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ we denote its transpose by $\mathbf{A}^T$, determinant by $\det(\mathbf{A})$ and trace by $Tr(\mathbf{A})$. If it is invertible, we denote the inverse by $\mathbf{A}^{-1}$. We let $\Lambda(\mathbf{A})$ denote the set of eigenvalues of $\mathbf{A}$. The p norm is denoted by $\|\cdot\|_p$ and defined as $\|\mathbf{x}\|_p = \{\sum_{i=1}^n |x^i|^p\}^{\frac{1}{p}}$.

Definition 1.2.1. *For $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n_+$ we write $\mathbf{x} \leq \mathbf{y}$ if $|x_i| \leq y_i$ for all $1 \leq i \leq n$. We write $\mathbf{x} \nleq \mathbf{y}$ otherwise.*

1.3 Brief Literature Review

This report is based on previous work in three main areas. We draw on results from quantization theory, networked control theory and stochastic control theory, particularly the study of Markov chains.

For an introduction to vector quantization, the reader is referred to [8] and [9]. The works [1] and [10] present a review of networked control theory and the relevant literature. A thorough treatment of Markov chains and stochastic stability can be found in [11].

In this report, we implement an adaptive quantizer in order to form a discrete estimate of the state. Adaptive quantizers change their bin sizes over time. They are outlined by Goodman and Gersho in [12], which studies an adaptive quantizer where the bin sizes expand or contract by a multiplicative constant at each time stage. It is shown that the logarithm of the range has a stationary distribution given an i.i.d. source. By choosing appropriate multiplicative constants M_i, it is shown that the state space of the quantizer range can be made irreducible. Such quantizers were developed for the purpose of speech encoding.

The paper [13] employs adaptive quantizers to show mean-stationarity of symbol-by-symbol encoding schemes when the source is stationary. It is shown that, under certain conditions, the application of the Goodman–Gersho adaptive quantization scheme results in the sequence of triples consisting of the bin size, input and output being asymptotically mean stationary [9]. Applications to variable-length coding schemes and queueing theory are also considered.

The programs considered in [12] and [13] cannot be applied to our setup since we are interested in systems that are open-loop unstable.

More recently, [14] studied the application of quantizers to both continuous and discrete linear time-invariant (LTI) control systems in the noise-free case. The approach taken yields global asymptotic stability. The class of quantizers with granular region $(-(M+1/2)\Delta, (M+1/2)\Delta]$, where Δ is the bin size and M is some positive integer is considered. The authors allow the bin size to change, expanding and contracting the granular region, and study the evolution of the bin size over time. Particular attention is paid to the case where M is small and the existence of stabilizing control policies are presented under various assumptions.

Since we consider multi-sensor and multi-controller systems, the decentralized control literature is also relevant. In decentralized control theory, [15] is an important work. It studies the effect of feedback in decentralized systems which are jointly observable and jointly controllable. The notion of a complete system is introduced. These are systems which can be made controllable and observable through the application of nondynamic decentralized feedback. Section 5 of [15] is of importance to us, as it applies standard techniques from graph theory to decompose a system into strongly connected components. We will employ a simplified version of such a decomposition when we study multi-sensor and multi-controller systems.

The work [16] studies decentralized control systems with LTI state equations and time-varying control laws. Assuming joint controllability and joint observability, it is shown that some systems which are unstabilizable via time-invariant control policies can be stabilized through the application of time-varying control policies. One of the main contributions of the paper is to show that many stability results can be obtained in the presence of fixed modes under linear time-invariant policies when the class of control laws is widened. Systems with noiseless plants and observations are studied.

Reference [5] considers a system with communication constraints. Here, we see that special attention is given to the development of coding and communication protocols. The work is motivated in part by remotely controlled systems in which delays are a central part of the analysis and performance is closely linked to the data rate of the control system. A major contribution of the article is the relation of containability, a weak form of stability, to the data rate.

References [5], [6] and [7] obtained a fundamental lower bound on the average rate of the information transmission for the finiteness of second moments. As derived in Section 1.1 for the system (1.1), this bound is

$$R_{\min} = \sum_{|\lambda_i|>1} \log_2(|\lambda_i|),$$

which holds for all linear systems in this report.

In [6], single-sensor, single-controller systems with communication constraints are studied. Motivated by geographically distributed systems, the paper states a control problem in which the sensor communicates with the controller via a discrete, noiseless channel. The goal is to design the channel encoder and decoder to achieve a variety of control objectives. Upper bounds are computed by explicitly stating the control policy.

The work [7] considered the case of finite-dimensional linear systems with both plant and observation noise. Stochastic stability is achieved for noise processes with unbounded support through the use of time-varying control laws. A variable-rate quantizer is used, in which the rate is very high at some time stages and zero at others. A new quantizer error bound is introduced and a lower bound on the data rate is derived.

Recent developments in decentralized systems under communication constraints include [17] and [18]. In [17], a noisy linear scalar system is considered in which quantized measurement are sent to the controller over a discrete noiseless channel. The existence of an invariant distribution and finite second moment are achieved for open-loop unstable systems using martingale theory. A fixed-rate coding and control policy is employed using adaptive quantizers.

The paper [18] uses random-time Lyapunov theory to obtain stability results for noisy scalar systems. In the setup of the paper, a coder communicates with a controller over an erasure channel with finite capacity. One contribution of the paper is the formulation of a general drift criteria which can be used to verify stability for Markov processes. The existence of a finite second moment for the state is established by obtaining a bound on the difference between stopping times.

In this report, we make use of the drift approach developed in [1], [17] and [18]. This method leads to greatly simplified proofs and allows us to develop an intuition about the stability of the systems considered. We are able to obtain strong forms of stability even in the presence of unbounded noise through the application of drift criteria.

In view of space constraints, we have only presented a summary of the directly related literature above and will simply refer the reader to additional material, e.g., [19], [20], [21] and [22].

1.4 Contributions

In view of the literature, the contributions of this work are as follows:

- The case where the system is multi-dimensional and driven by unbounded noise over a discrete-channel has not been studied to our knowledge, regarding the existence of an invariant distribution and ergodicity properties. Results for the limit properties of the finite moment are also new.

- We give sufficient conditions for multi-sensor and multi-controller systems with both system noise and observation noise with unbounded support, which has not been treated previously, to our knowledge.

Our approach builds on martingales and random-drift programs, as considered in [17] and [18]. However, new geometric constructions are needed for the vector and partially observed settings. We define a more general class of stopping times and adopt a further geometric approach than what is present in these papers.

We structure this report as follows. In Chapter 2, we study single-station systems driven by Gaussian noise and give our main result for such systems, Theorem 2.2.1. Chapter 3 extends these results to a larger class of noise distributions with sufficiently light tails. Chapters 4 and 5 study the multi-sensor (single-controller) and multi-controller (single-sensor) cases respectively. We give concluding remarks in Chapter 6 and suggestions for further research in Chapter 7. Some basic definitions and results from the theory of matrices, Markov chains and stochastic stabilization are provided in the Appendix.

Chapter 2

Single-Station Systems

In this chapter, we consider single-sensor, single-controller systems.

2.1 Problem Statement

Consider the class of single-station LTI discrete-time systems with both plant and observation noise. The system equations are given by

$$\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{B}\mathbf{u}_t + \mathbf{w}_t, \quad \mathbf{y}_t = \mathbf{C}\mathbf{x}_t + \mathbf{v}_t, \tag{2.1}$$

where $\mathbf{x}_t \in \mathbb{R}^n$, $\mathbf{u}_t \in \mathbb{R}^m$ and $\mathbf{y}_t \in \mathbb{R}^p$ are the state, control action and observation at time t respectively. The matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and the noise vectors $\mathbf{w}_t, \mathbf{v}_t$ are of compatible size. The initial state, $\mathbf{x}_0$, is drawn from a Gaussian distribution.

Assumption 2.1.1. *The noise processes $\{\mathbf{w}_t\}$ and $\{\mathbf{v}_t\}$ are each i.i.d. sequences of multivariate Gaussian random vectors with zero mean. At time t, both $\mathbf{w}_t$ and $\mathbf{v}_t$ are independent of $\mathbf{x}_t$ and each other.*

Assumption 2.1.2. *The pair $(\mathbf{A}, \mathbf{B})$ is controllable and the pair $(\mathbf{C}, \mathbf{A})$ is observable.*

The setup is depicted in Figure 2.1. The observations are made by the sensor and sent to the controller through a finite capacity channel. At each time stage t, we allow the sensor to send an encoded value $q_t \in \{1, \ldots, N_t\}$ for some $N_t \in \mathbb{N}$. We define the rate of our system at time t as $R_t = \log_2(N_t)$. Now, suppose that the channel is used periodically, every T time stages. The rate for all time stages is then specified by $\{N_0, \ldots, N_{T-1}\}$. The average rate is

$$R_{\text{avg}} = \frac{1}{T} \sum_{t=0}^{T-1} R_t. \tag{2.2}$$

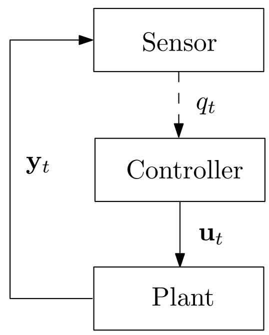

Figure 2.1: A single-station system with finite-rate communication channel.

Information structure. For a process $\{\mathbf{x}_t\}$ we define $\mathbf{x}_{[a,b]} = \{\mathbf{x}_a, \mathbf{x}_{a+1}, \ldots, \mathbf{x}_b\}$. At time t, the sensor maps its information $I_t^s := \mathbf{y}_{[0,t]} \rightarrow q_t \in \{1, \ldots, N_t\}$. The controller maps its information $I_t^c := q_{[0,t]} \rightarrow \mathbf{u}_t \in \mathbb{R}^m$.

2.2 Main Result

We label the eigenvalues of $\mathbf{A}$ as $\lambda_1, \ldots, \lambda_n$. Without loss of generality, we assume that $|\lambda_i| > 1$ for all $1 \leq i \leq n$. Our main result for single-station systems is the following:

Theorem 2.2.1. *There exists a coding and control policy with average rate* $R_{avg} \leq 1/(T2n) \sum_{i=1}^{n} \log_2(\lceil |\lambda_i|^{T2n} + \epsilon \rceil + 1)$ *for some* $\epsilon > 0$ *which gives:*

(a) the existence of a unique invariant distribution for $\{\mathbf{x}_{2nt}\}$*;*

(b) $\lim_{t\rightarrow\infty} E[\|\mathbf{x}_{2nt}\|_2^2] < \infty$.

Theorem 2.2.2. *The average rate in Theorem 2.2.1 achieves the minimum rate* (1.9) *asymptotically for large sampling periods. That is,* $\lim_{T\rightarrow\infty} R_{avg} = R_{\min}$.

2.3 Coding and Control Policy

For now, assume that $\mathbf{A}$ has only one eigenvalue λ. See Remark 2.4.2 for a justification of this. We will give an explicit coding and control policy in a more general setting in Chapters 4 and 5.

Put $K = \lceil |\lambda| + \epsilon \rceil$ for some parameter $\epsilon > 0$ and consider the following scalar $(K+1)$-bin uniform quantizer. Assuming that K is even, this is defined for $k \in \{1, 2, \ldots, K\}$ as

$$Q_K^\Delta(x) = \begin{cases} \left(-\frac{K+1}{2} + k\right)\Delta, & \text{if } x \in \left[\left(\frac{-K}{2} + k - 1\right)\Delta, \left(\frac{-K}{2} + k\right)\Delta\right), \\ \frac{K-1}{2}\Delta, & \text{if } |x| = \frac{K}{2}\Delta, \\ 0, & \text{if } |x| > \frac{K}{2}\Delta, \end{cases}$$

where $\Delta \in \mathbb{R}_+$ is the bin size. The set $[-\frac{K}{2}\Delta, \frac{K}{2}\Delta]$ is called the granular region while the set $(-\infty, -\frac{K}{2}\Delta) \cup (\frac{K}{2}\Delta, \infty)$ is called the overflow region. If the state is in the granular region, that is if $|x| \leq \frac{K}{2}\Delta$, then we say the quantizer is *perfectly-zoomed* at x. Otherwise, we say it is *under-zoomed*.

We write our quantizer as the composite function $Q_K^\Delta(x) = \mathcal{D}_K^\Delta(\mathcal{E}_K^\Delta(x))$. The encoder $\mathcal{E}_K^\Delta : \mathbb{R} \to \{0, 1, \ldots, K\}$ and decoder $\mathcal{D}_K^\Delta : \{0, 1, \ldots, K\} \to \mathcal{C}$ for $k \in \{1, 2, \ldots, K\}$ are

$$\mathcal{E}_K^\Delta(x) = \begin{cases} k, & \text{if } x \in [(\frac{-K}{2} + k - 1)\Delta, (\frac{-K}{2} + k)\Delta), \\ K, & \text{if } x = \frac{K}{2}\Delta, \\ 0, & \text{if } |x| > \frac{K}{2}\Delta, \end{cases}$$

$$\mathcal{D}_K^\Delta(x) = \begin{cases} \left(-\frac{K+1}{2} + x\right)\Delta, & \text{if } x \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

At time t, we associate with each component x_t^i a bin size Δ_t^i. Let $q_t^i = \mathcal{E}_K^{\Delta_t^i}(y_t^i)$. We will be applying our control policy to system (2.6) where $\mathbf{y}_s$ is a meaningful estimate of the state $\mathbf{x}_s$. Let $N_t = K^n + 1$ for all $t \in \mathbb{N}$. Choose any invertible function

$f : \{1, \dots, K\}^n \to \{1, \dots, K^n\}$. We then choose the encoded value

$$q_t = \begin{cases} f(q_t^1, \dots, q_t^n), & \text{if } q_t^i \neq 0 \text{ for all } 1 \leq i \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Upon receiving $q_t \neq 0$, the controller knows $q_t^1, \dots, q_t^n$. The controller forms the estimate $\hat{\mathbf{x}}_t$ as $\hat{\mathbf{x}}_t = \begin{bmatrix} \hat{x}_t^1 & \cdots & \hat{x}_t^n \end{bmatrix}^T$, where

$$\hat{x}_t^i = \begin{cases} \mathcal{D}_K^{\Delta_t^i}(q_t^i), & \text{if } q_t \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

We assume without loss of generality that $\mathbf{A}$ is a Jordan block with eigenvalue λ. From the real Jordan canonical form (see for example [23]), we know that it can be written as

$$\mathbf{A} = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}, \quad \text{if } \lambda \in \mathbb{R}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{D} & \mathbf{I} & & \\ & \mathbf{D} & \ddots & \\ & & \ddots & \mathbf{I} \\ & & & \mathbf{D} \end{bmatrix}, \quad \text{if } \lambda \in \mathbb{C},$$

where in the complex case we write $\lambda = a + ib$ for some $a, b \in \mathbb{R}$ and define

$$\mathbf{D} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

The update equations are

$$\Delta_{t+1} = \bar{Q}\left(q_t, \Delta_t\right)\Delta_t, \quad \bar{Q}\left(q_t, \Delta_t\right) = \begin{cases} \rho|\lambda|, & \text{if } q_t = 0, \\ \beta(\Delta_t), & \text{otherwise,} \end{cases} \tag{2.3}$$

for some $\rho > 1$ and with

$$\beta(\Delta_t) = \mathrm{diag}(\beta_1(\Delta_t^1), \ldots, \beta_n(\Delta_t^n)), \quad \beta_i(\Delta_t^i) = \begin{cases} 1, & \text{if } \Delta_t^i \leq L^i \ , \\ \frac{|\lambda|}{|\lambda|+\epsilon-\eta}, & \text{otherwise,} \end{cases} \tag{2.4}$$

for some $0 < \eta < \epsilon$ and $\mathbf{L} \in \mathbb{R}_+^n$. Note that if we define $\bar{\mathbf{L}} = \mathbf{L}|\lambda|/(|\lambda| + \epsilon - \eta)$ then $\Delta_t^i > \bar{L}^i$ for all $1 \leq i \leq n$ and all $t \in \mathbb{N}$.

Bin ordering. We set $\mathbf{L} = c\Delta_0$, for some $0 < c \leq 1$. First let $\lambda \in \mathbb{R}$. For any $\delta > 0$ we can choose Δ_0^i and Δ_0^{i+1} such that $\Delta_0^{i+1} \leq \delta\Delta_0^i$ for all $1 \leq i \leq n-1$. With our update equations and our choice of $\mathbf{L}$ we get that the ordering is preserved over all time stages. That is, $\Delta_t^{i+1} \leq \delta\Delta_t^i$ for all $1 \leq i \leq n-1$ and $t \in \mathbb{N}$.

Now let $\lambda \in \mathbb{C}$. We choose $\Delta_0^i = \Delta_0^{i+1}$ for all i odd. Thus, we have divided the complex modes into their conjugate pairs and set their initial bin sizes to be equal. Our initial condition implies that $\Delta_t^i = \Delta_t^{i+1}$ for all i odd and $t \in \mathbb{N}$. For any $\delta > 0$ we can choose Δ_0^i and Δ_0^{i+2} such that $\Delta_t^{i+2} \leq \delta\Delta_t^i$ for all $1 \leq i \leq n-2$ and $t \in \mathbb{N}$.

Control action. Under our information structure, the update equations (2.3) can be applied at the sensor and the controller. Our vector quantizer is implementable and at time t the controller knows $\hat{\mathbf{x}}_t$. We choose the control action

$$\mathbf{u}_t = -\mathbf{A}\hat{\mathbf{x}}_t.$$

2.4 Outline of Proof for Theorem 2.2.1

In this section, we outline the supporting results and key steps in proving our main result for single-station systems, Theorem 2.2.1. The proofs are given in Section 2.5.

Lemma 2.4.1. *We can sample every $2n$ time stages and apply a similarity transform to $\mathbf{x}_t$ in (2.1) to obtain $\bar{\mathbf{x}}_s = \mathbf{P}\mathbf{x}_{2ns}$ with $s \in \mathbb{N}$ for some invertible matrix $\mathbf{P}$. This new state satisfies the following system of equations:*

$$\bar{\mathbf{x}}_{s+1} = \bar{\mathbf{A}}\bar{\mathbf{x}}_s + \bar{\mathbf{u}}_s + \bar{\mathbf{w}}_s, \quad \bar{\mathbf{y}}_s = \bar{\mathbf{x}}_s + \bar{\mathbf{v}}_s. \tag{2.5}$$

The control action $\bar{\mathbf{u}}_s \in \mathbb{R}^n$ is chosen arbitrarily by the controller. The estimate $\bar{\mathbf{y}}_s \in \mathbb{R}^n$ at time s is known by the sensor. The noise processes $\{\bar{\mathbf{w}}_s\}$ and $\{\bar{\mathbf{v}}_s\}$ are *each i.i.d. sequences of zero mean multivariate Gaussian random vectors. At time* s*,* $\bar{\mathbf{w}}_s$ *and* $\bar{\mathbf{v}}_s$ *are independent of* $\bar{\mathbf{x}}_s$ *but may be correlated with each other. For* $s_1 \neq s_2$*, the vectors* $\bar{\mathbf{w}}_{s_1}$ *and* $\bar{\mathbf{v}}_{s_2}$ *are independent. The matrix* $\bar{\mathbf{A}}$ *is in real Jordan normal form and has eigenvalues* $\lambda_1^{2n}, \ldots, \lambda_n^{2n}$*.*

By a slight abuse of notation, we will rewrite system (2.5) as

$$\mathbf{x}_{s+1} = \mathbf{A}\mathbf{x}_s + \mathbf{u}_s + \mathbf{w}_s, \quad \mathbf{y}_s = \mathbf{x}_s + \mathbf{v}_s, \tag{2.6}$$

where $\mathbf{x}_s \in \mathbb{R}^n$, $\mathbf{u}_s \in \mathbb{R}^n$ and $\mathbf{y}_s \in \mathbb{R}^n$ are the state, control action and observation at time s respectively.

Remark 2.4.2. *We consider the case where* $\mathbf{A}$ *is a single Jordan block with eigenvalue* λ*. We can do this without loss of generality since we are considering the single-station case and the sensor obtains an estimate for all components, as seen in Lemma 2.4.1. Thus, we can simply apply our control policy to each Jordan block. In all remaining theorems of this section, we will work with system* (2.6)*. Where necessary, we will distinguish between the real and complex eigenvalue cases.*

Lemma 2.4.3. *The process* $\{(\mathbf{x}_s, \Delta_s)\}$ *is Markov.*

Section 2.3 gives our control policy in terms of the parameters ρ, ϵ and η.

Lemma 2.4.4. *For appropriate choices of* ρ, ϵ *and* η*, we can form a countable state space* $\mathcal{S}$ *for* $\{\Delta_s\}$*. The process* $\{(\mathbf{x}_s, \Delta_s)\}$ *is an irreducible and aperiodic Markov chain on* $\mathbb{R}^n \times \mathcal{S}$*.*

Define the sequence of stopping times

$$\tau_0 = 0, \quad \tau_{z+1} = \min\left\{ s > \tau_z : |\mathbf{y}_s| = |\mathbf{x}_s + \mathbf{v}_s| \leq \frac{K}{2}\Delta_s \right\}.$$

These are the times when all quantizers are perfectly-zoomed. We assume that this is satisfied at time $s = 0$. This technical condition is justified by showing that the process $\{(\mathbf{x}_s, \Delta_s)\}$ moves to such a perfectly zoomed state in a finite time, which is dominated by a geometric distribution (see the proof of Proposition 3.2 in [18]).

Theorem 2.4.5. *If* K *is even then the following hold.*

(a) *For any $r > 1$ and any polynomial $Q(k)$ of finite degree there exists a sufficiently large H such that $Q(k)P(\tau_{z+1} - \tau_z > k \mid \mathbf{x}_{\tau_z}, \Delta_{\tau_z}) \leq r^{-k}$ for all $k > H$, for all $z \in \mathbb{N}$ and for all $(\mathbf{x}_{\tau_z}, \Delta_{\tau_z})$.*

(b) *Let $\Delta_{\tau_z} \to \infty$ be equivalent to stating that $\Delta^i_{\tau_z} \to \infty$ for all $1 \leq i \leq n$. Then*

$$\lim_{\Delta_{\tau_z} \to \infty} P(\tau_{z+1} - \tau_z > 1 \mid \mathbf{x}_{\tau_z}, \Delta_{\tau_z}) = 0$$

uniformly in $\mathbf{x}_{\tau_z}$.

We define the compact sets

$$S = S_{\mathbf{x}} \times S_{\Delta}, \quad S_{\Delta} = \{\Delta \in \mathbb{R}^n_+ : \Delta^i \leq F, 1 \leq i \leq n\},$$
$$S_{\mathbf{x}} = \left\{ \mathbf{x} \in \mathbb{R}^n : |x^i| \leq \frac{K}{2} F, 1 \leq i \leq n \right\},$$

for some $F > L^1$ where L^1 is a component of $\mathbf{L}$ as described in Section 2.3. Note that at the stopping time τ_z, if $\Delta_{\tau_z} \in S_{\Delta}$ then $|x^i_{\tau_z}| \leq \frac{K}{2}\Delta^i_{\tau_z} \leq \frac{K}{2}F$, for all $1 \leq i \leq n$, and thus $\mathbf{x}_{\tau_z} \in S_{\mathbf{x}}$ and $(\mathbf{x}_{\tau_z}, \Delta_{\tau_z}) \in S$.

Lemma 2.4.6. *For some $\gamma > 0$, the following drift condition holds:*

$$\gamma E\left[\sum_{s=\tau_z}^{\tau_{z+1}-1} (\Delta^1_s)^2 \middle| \mathbf{x}_{\tau_z}, \Delta_{\tau_z} \right] \leq (\Delta^1_{\tau_z})^2 - E[(\Delta^1_{\tau_{z+1}})^2 \mid \mathbf{x}_{\tau_z}, \Delta_{\tau_z}] + b1_{\{(\mathbf{x}_{\tau_z}, \Delta_{\tau_z}) \in S\}}. \quad (2.7)$$

For $\lambda \in \mathbb{C}$, the above also holds with Δ^2 in place of Δ^1.

For $\mathbf{x} \in \mathbb{R}^n$, we say that x^i and x^{i+1} are a *conjugate pair* if i is odd. To simplify notation in the complex eigenvalue case we find it convenient to define for any $\mathbf{x} \in \mathbb{R}^n$, the set of vectors

$$\tilde{\mathbf{x}}^i = \begin{bmatrix} x^i & x^{i+1} \end{bmatrix}^T, \quad \text{if } i \text{ is odd}, \qquad \tilde{\mathbf{x}}^i = \begin{bmatrix} x^{i-1} & x^i \end{bmatrix}^T, \quad \text{if } i \text{ is even},$$

for $1 \leq i \leq n$. Note that $\tilde{\mathbf{x}}^i = \tilde{\mathbf{x}}^{i+1}$ for i odd. We are only concerned with the case when n is even.

Theorem 2.4.7. *Let $\lambda \in \mathbb{R}$. For $i = n$, there exists a $\kappa > 0$ such that*

$$E\left[\sum_{s=\tau_z}^{\tau_{z+1}-1} (x_s^i)^2 \middle| \mathbf{x}_{\tau_z}, \Delta_{\tau_z}\right] \leq \kappa(\Delta_{\tau_z}^1)^2. \tag{2.8}$$

If $\lim_{s\to\infty} E[(x_s^k)^2] < \infty$ then the above holds for $i = k - 1$.

For $\lambda \in \mathbb{C}$, with $i = n - 1$, there exists a $\kappa > 0$ such that

$$E\left[\sum_{s=\tau_z}^{\tau_{z+1}-1} (\tilde{\mathbf{x}}_s^i)^T \tilde{\mathbf{x}}_s^i \middle| \mathbf{x}_{\tau_z}, \Delta_{\tau_z}\right] \leq \kappa(\tilde{\Delta}_{\tau_z}^1)^T \tilde{\Delta}_{\tau_z}^1.$$

If $\lim_{s\to\infty} E[(\tilde{\mathbf{x}}_s^k)^T \tilde{\mathbf{x}}_s^k] < \infty$ then the above holds for $i = k - 2$.

In both cases, we have that κ does not depend on the condition $(\mathbf{x}_{\tau_z}, \Delta_{\tau_z})$.

Proof of Theorem 2.2.1:

(a) We know from Lemmas 2.4.3 and 2.4.4 that the process $\{(\mathbf{x}_s, \Delta_s)\}$ is an irreducible and aperiodic Markov chain. The set S is small (see the Appendix and [18]). Using Lemma 2.4.6 we can apply Theorem B.0.8 with $a = 1$, the Markov chain $\{(\mathbf{x}_s, \Delta_s)\}$ and the functions $V(\mathbf{x}_s, \Delta_s) = (\Delta_s^1)^2$, $\beta(\mathbf{x}_s, \Delta_s) = 1$ and b as given in Lemma 2.4.6 to get that $\{(\mathbf{x}_s, \Delta_s)\}$ is positive Harris recurrent and has a unique invariant distribution. Note that this proof holds since γ has no dependence on Δ_{τ_z}. We choose $\mathbf{L}$ large enough such that we can pick ζ in Theorem 2.4.6. We then fix ζ, γ and expand $\mathbf{L}$ further, if necessary, to get $\gamma\Delta_t > 1$ for all t.

(b) Suppose that $\lambda \in \mathbb{R}$. We will apply Theorem B.0.8 with $a = 0$, the Markov chain $\{(\mathbf{x}_s, \Delta_s)\}$ and the functions

$$V(\mathbf{x}_s, \Delta_s) = (\Delta_s^1)^2, \beta(\mathbf{x}_s, \Delta_s) = \gamma(\Delta_s^1)^2, f(\mathbf{x}_s, \Delta_s) = \frac{\gamma}{\kappa}(x_s^n)^2.$$

From Lemma 2.4.6, we get

$$\begin{aligned}
&E[V(\mathbf{x}_{\tau_{z+1}}, \Delta_{\tau_{z+1}}) \mid \mathcal{F}_{\tau_z}] = E[(\Delta_{\tau_{z+1}}^1)^2 \mid \mathbf{x}_{\tau_z}, \Delta_{\tau_z}] \\
&\leq (\Delta_{\tau_z}^1)^2 - \gamma E\left[\sum_{s=\tau_z}^{\tau_{z+1}-1} (\Delta_s^1)^2 \middle| \mathbf{x}_{\tau_z}, \Delta_{\tau_z}\right] + b1_{\{(\mathbf{x}_{\tau_z}, \Delta_{\tau_z}) \in S\}}
\end{aligned}$$

$$\leq (\Delta^1_{\tau_z})^2 - \gamma(\Delta^1_{\tau_z})^2 + b1_{\{(\mathbf{x}_{\tau_z},\Delta_{\tau_z})\in S\}}$$
$$= V(\mathbf{x}_{\tau_z}, \Delta_{\tau_z}) - \beta(\mathbf{x}_{\tau_z}, \Delta_{\tau_z}) + b1_{\{(\mathbf{x}_{\tau_z},\Delta_{\tau_z})\in S\}}.$$

We know that Theorem 2.4.7 holds immediately for $\{x^n_s\}$ and thus

$$E\left[\sum_{s=\tau_z}^{\tau_{z+1}-1} f(\mathbf{x}_s, \Delta_s)\middle| \mathcal{F}_{\tau_z}\right] = \frac{\gamma}{\kappa} E\left[\sum_{s=\tau_z}^{\tau_{z+1}-1} (x^n_s)^2 \middle| \mathbf{x}_{\tau_z}, \Delta_{\tau_z}\right] \leq \gamma(\Delta^1_{\tau_z})^2 = \beta(\mathbf{x}_{\tau_z}, \Delta_{\tau_z}).$$

Thus, $\lim_{s\to\infty} \frac{\gamma}{\kappa} E[(x^n_s)^2] < \infty$ by Theorem B.0.8 and so $\lim_{s\to\infty} E[(x^n_s)^2] < \infty$. This implies that Theorem 2.4.7 holds for $\{x^{n-1}_s\}$ as mentioned in the proof and theorem statement. The finite second moment of all components then follows by induction.

In the complex case, we have that the drift condition (2.7) in Lemma 2.4.6 also holds with Δ^2_s in place of Δ^1_s since they are equal. Choosing the functions $V(\mathbf{x}_s, \Delta_s) = (\Delta^1_s)^2 + (\Delta^2_s)^2$, $\beta(\mathbf{x}_s, \Delta_s) = \gamma((\Delta^1_s)^2 + (\Delta^2_s)^2)$, $f(\mathbf{x}_s, \Delta_s) = \frac{\gamma}{\kappa}(\tilde{\mathbf{x}}^n_s)^T \tilde{\mathbf{x}}^n_s$, we obtain the result.

□

Finally, a remark on the policy we have employed in this chapter is in order.

Remark 2.4.8. *We have presented a vector stabilization scheme. From the problem statement, it would be natural to adopt a sequential stabilization scheme. That is, each of the components of the state is viewed as a separate system. In this case, we lose the Markov property and the number of time stages we must wait (denoted by H in Theorem 2.4.5) to establish geometric decay is dependent on the conditions $(\mathbf{x}_{\tau_z}, \Delta_{\tau_z})$. This complicates the analysis and such a scheme is left for future work.*

2.5 Supporting Results for Section 2.4

Proof of Theorem 2.2.2: In Sections 2.3 and 2.4, we describe our control policy for period $T = 2n$ with a fixed average rate of $R_{\text{avg}} = \frac{1}{2n}\log_2\left(\{\prod_{i=1}^n \lceil|\lambda_i|^{2n} + \epsilon\rceil\} + 1\right)$. Suppose that instead of sending an estimate every $2n$ time stages, we apply them periodically every $T2n$ time stages. Taking the limit as T approaches infinity, our average rate satisfies

$$\lim_{T\to\infty} R_{avg} \leq \lim_{T\to\infty} \frac{1}{T2n}\left(\sum_{i=1}^{n} \log_2(\lceil|\lambda_i|^{T2n}+\epsilon\rceil+1)\right)$$
$$= \lim_{T\to\infty}\left(\sum_{i=1}^{n} \log_2(\lceil|\lambda_i|^{T2n}+\epsilon\rceil+1)^{\frac{1}{T2n}}\right) = \sum_{i=1}^{n}\log_2(|\lambda_i|).$$

In this sense, our policy achieves the minimum rate (1.9) asymptotically. □

Proof of Lemma 2.4.1: Recall the basic recursion for LTI systems.

$$\mathbf{x}_t = \mathbf{A}\mathbf{x}_{t-1} + \mathbf{B}\mathbf{u}_{t-1} + \mathbf{w}_{t-1} = \mathbf{A}^2\mathbf{x}_{t-2} + \mathbf{A}\mathbf{B}\mathbf{u}_{t-2} + \mathbf{B}\mathbf{u}_{t-1} + \mathbf{A}\mathbf{w}_{t-2} + \mathbf{w}_{t-1}$$
$$\cdots = \mathbf{A}^t\mathbf{x}_0 + \sum_{i=0}^{t-1}\mathbf{A}^{t-1-i}\mathbf{B}\mathbf{u}_i + \sum_{i=0}^{t-1}\mathbf{A}^{t-1-i}\mathbf{w}_i.$$

In the first n time stages the sensor makes observations on the state and forms an estimate. In the second n time stages we allow the controller to apply a control action.

We set $\mathbf{u}_i = 0$ for $0 \leq i \leq n-2$ so that the first n observations of the sensor are

$$\begin{bmatrix} \mathbf{y}_0 \\ \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_{n-1} \end{bmatrix} = \mathcal{O}_{(\mathbf{C},\mathbf{A})}\mathbf{x}_0 + \begin{bmatrix} 0 \\ \mathbf{C}\mathbf{w}_0 \\ \vdots \\ \sum_{i=0}^{n-2}\mathbf{C}\mathbf{A}^{n-2-i}\mathbf{w}_i \end{bmatrix} + \begin{bmatrix} \mathbf{v}_0 \\ \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_{n-1} \end{bmatrix},$$

where $\mathcal{O}_{(\mathbf{C},\mathbf{A})}$ is the observability matrix of the pair $(\mathbf{C},\mathbf{A})$. We have assumed that $(\mathbf{C},\mathbf{A})$ is an observable pair as part of Assumption 2.1.2. Equivalently, $\mathcal{O}_{(\mathbf{C},\mathbf{A})}$ has full column rank. By choosing a subset of n equations from the matrix equation above, it is clear that we can apply the inverse to obtain the estimate

$$\hat{\mathbf{y}}_0 = \mathbf{x}_0 + \sum_{i=0}^{n-2}\xi_i\mathbf{w}_i + \sum_{i=0}^{n-1}\zeta_i\mathbf{v}_i,$$

for some set $\{\xi_i, \zeta_i\}$ of matrices $\xi_i \in \mathbb{R}^{n\times n}$ and $\zeta_i \in \mathbb{R}^{n\times p}$.

Our estimate $\hat{\mathbf{y}}_0$ is generated at time $n-1$. At this time stage, the sensor sends the encoded value q_{n-1} to the controller through the finite capacity channel. Based on this information, we allow the controller to apply control actions in time stages n to $2n-1$. This is standard (see for example [24]) and we do not describe it in detail. We then have the system of equations

$$\mathbf{x}_{2n} = \mathbf{A}^{2n}\mathbf{x}_0 + \tilde{\mathbf{u}}_0 + \sum_{i=0}^{2n-1} \mathbf{A}^{2n-1-i}\mathbf{w}_i, \quad \hat{\mathbf{y}}_0 = \mathbf{x}_0 + \sum_{i=0}^{n-2} \xi_i \mathbf{w}_i + \sum_{i=0}^{n-1} \zeta_i \mathbf{v}_i,$$

where at time $n-1$, the estimate $\hat{\mathbf{y}}_0$ is known by the sensor and the action $\tilde{\mathbf{u}}_0$ is chosen arbitrarily by the controller.

Let us define the sampled variables $\tilde{\mathbf{x}}_s = \mathbf{x}_{2ns}$ and $\tilde{\mathbf{y}}_s = \hat{\mathbf{y}}_{2ns}$. We define the noise processes

$$\tilde{\mathbf{w}}_s = \sum_{i=0}^{2n-1} \mathbf{A}^{2n-1-i}\mathbf{w}_{2ns+i}, \quad \tilde{\mathbf{v}}_s = \sum_{i=0}^{n-2} \xi_i \mathbf{w}_{2ns+i} + \sum_{i=0}^{n-1} \zeta_i \mathbf{v}_{2ns+i},$$

and note that they are both sequences of i.i.d. multivariate Gaussian random vectors with zero mean. Then, by repeating our procedure every $2n$ time stages, we obtain the system

$$\tilde{\mathbf{x}}_{s+1} = \mathbf{A}^{2n}\tilde{\mathbf{x}}_s + \tilde{\mathbf{u}}_s + \tilde{\mathbf{w}}_s, \quad \tilde{\mathbf{y}}_s = \tilde{\mathbf{x}}_s + \tilde{\mathbf{v}}_s.$$

Finally, we apply a real Jordan transformation to the above system. We define $\bar{\mathbf{x}}_s = \mathbf{P}\tilde{\mathbf{x}}_s$, $\bar{\mathbf{A}} = \mathbf{P}\mathbf{A}^{2n}\mathbf{P}^{-1}$, $\bar{\mathbf{u}}_s = \mathbf{P}\tilde{\mathbf{u}}_s$, $\bar{\mathbf{w}}_s = \mathbf{P}\tilde{\mathbf{w}}_s$, $\bar{\mathbf{y}}_s = \mathbf{P}\tilde{\mathbf{y}}_s$ and $\bar{\mathbf{v}}_s = \mathbf{P}\tilde{\mathbf{v}}_s$ where $\mathbf{P}$ is the Jordan transform matrix. This gives the system

$$\bar{\mathbf{x}}_{s+1} = \bar{\mathbf{A}}\bar{\mathbf{x}}_s + \bar{\mathbf{u}}_s + \bar{\mathbf{w}}_s, \quad \bar{\mathbf{y}}_s = \bar{\mathbf{x}}_s + \bar{\mathbf{v}}_s.$$

Note that the matrix $\bar{\mathbf{A}}$ has eigenvalues $\lambda_1^{2n}, \ldots, \lambda_n^{2n}$. □

Remark 2.5.1. *The estimate used in Lemma 2.4.1 may appear naive. At first glance, it would seem better to apply the Kalman filter. In this case, a new system is formed with the estimate as the state. The problem is that the noise for this system is not independent across time. We must obtain a new bound on the noise using the stability of the Kalman filter. That is, using the fact that the covariance matrix of the noise process converges.*

Furthermore, the noise is orthogonal to the state in the sense that they are uncorrelated. In the Gaussian case, the state and noise are thus independent, but this does not hold for a more general class of noise distributions and we lose the Markov property.

Proof of Lemma 2.4.3: Note that under our control policy we can write $\mathbf{u}_s = g(\mathbf{x}_s, \mathbf{v}_s, \Delta_s)$ and $\Delta_{s+1} = f(\mathbf{x}_s, \mathbf{v}_s, \Delta_s)$ for some functions g and f.

Let $\mathcal{B}(\mathbb{R}^n \times \mathbb{R}^n_+)$ be the Borel σ-field on $\mathbb{R}^n \times \mathbb{R}^n_+$. It follows that

$$
\begin{aligned}
& P\left((\mathbf{x}_{s+1}, \Delta_{s+1}) \in (C \times D) \mid (\mathbf{x}_s, \Delta_s), \ldots, (\mathbf{x}_0, \Delta_0)\right) \\
&= P\left(\mathbf{x}_{s+1} \in C \mid \Delta_{s+1} \in D, (\mathbf{x}_s, \Delta_s), \ldots, (\mathbf{x}_0, \Delta_0)\right) \\
&\qquad P\left(\Delta_{s+1} \in D \mid (\mathbf{x}_s, \Delta_s), \ldots, (\mathbf{x}_0, \Delta_0)\right) \\
&= P\left(\mathbf{A}\mathbf{x}_s + \mathbf{u}_s + \mathbf{w}_s \in C \mid \Delta_{s+1} \in D, (\mathbf{x}_s, \Delta_s), \ldots, (\mathbf{x}_0, \Delta_0)\right) \\
&\qquad P\left(f(\mathbf{x}_s, \mathbf{v}_s, \Delta_s) \in D \mid (\mathbf{x}_s, \Delta_s), \ldots, (\mathbf{x}_0, \Delta_0)\right) \\
&= P\left(\mathbf{A}\mathbf{x}_s + g(\mathbf{x}_s, \mathbf{v}_s, \Delta_s) + \mathbf{w}_s \in C \mid \Delta_{s+1} \in D, (\mathbf{x}_s, \Delta_s), \ldots, (\mathbf{x}_0, \Delta_0)\right) \\
&\qquad P\left(f(\mathbf{x}_s, \mathbf{v}_s, \Delta_s) \in D \mid (\mathbf{x}_s, \Delta_s), \ldots, (\mathbf{x}_0, \Delta_0)\right) \\
&= P\left(\mathbf{A}\mathbf{x}_s + g(\mathbf{x}_s, \mathbf{v}_s, \Delta_s) + \mathbf{w}_s \in C \mid \Delta_{s+1} \in D, (\mathbf{x}_s, \Delta_s)\right) \\
&\qquad P\left(f(\mathbf{x}_s, \mathbf{v}_s, \Delta_s) \in D \mid (\mathbf{x}_s, \Delta_s)\right) \\
&= P\left((\mathbf{x}_{s+1}, \Delta_{s+1}) \in (C \times D) \mid (\mathbf{x}_s, \Delta_s)\right),
\end{aligned}
$$

for all $(C \times D) \in \mathcal{B}(\mathbb{R}^n \times \mathbb{R}^n_+)$. □

Proof of Lemma 2.4.4: This follows immediately from the scalar case, as presented in the proof of Theorem 2.4 of [17]. We can choose ρ, ϵ and η such that $\log_2(\bar{Q}^i(q_s, \Delta_s))$, where $\bar{Q}^i(q_s, \Delta_s) \in \{\rho|\lambda|, \beta_i(\Delta^i_s)\}$ is the i^{th} component of $\bar{Q}(q_s, \Delta_s)$, takes values in integer multiples of ℓ and the integers taken are relatively prime. By setting each Δ^i_0 to be an integer multiple of ℓ, it follows from the equation

$$\log_2(\Delta^i_{s+1})/\ell = \log_2(\bar{Q}^i(q_s, \Delta_s))/\ell + \log_2(\Delta^i_s)/\ell$$

that $\log_2(\Delta^i_s)$ is an integer multiple of ℓ for all $s \in \mathbb{N}$. The above construction, coupled with the assumption that the noise process $\{\mathbf{w}_s\}$ is drawn from a distribution which is positive on every open set, gives irreducibility.

As is [17], the chain is aperiodic since the bin sizes can take their smallest value for any finite number of consecutive time stages with positive probability. □

To prove Theorem 2.4.5, we need the following simple Gaussian bound. Recall that $\Lambda(\cdot)$ denotes the set of eigenvalues of its argument. Let us define $\lambda_{\min}(\mathbf{A}) = \min \Lambda(\mathbf{A})$ and $\lambda_{\max}(\mathbf{A}) = \max \Lambda(\mathbf{A})$.

Lemma 2.5.2. *Let $\mathbf{X} \sim \mathcal{N}(0, \mathbf{\Sigma})$ be a multivariate normal random vector with mean zero and covariance matrix $\mathbf{\Sigma} \in \mathbb{R}^{n \times n}$. For $\Delta \in \mathbb{R}^n_+$, the following bound holds.*

$$P(\mathbf{X} \not\leq \Delta) \leq 2\sqrt{\frac{\lambda^{n+1}_{\max}(\mathbf{\Sigma})}{2\pi \det(\mathbf{\Sigma})}} \sum_{i=1}^{n} \exp\left\{-\frac{(\Delta^i)^2}{2\lambda_{\max}(\mathbf{\Sigma})}\right\}.$$

Proof of Lemma 2.5.2: Let $\mathbf{X} \sim \mathcal{N}(0, \mathbf{\Sigma})$ be a multivariate normal random vector with mean zero and covariance matrix $\mathbf{\Sigma} \in \mathbb{R}^{n \times n}$. We avoid the degenerate case and assume that $\mathbf{\Sigma}$ is positive-definite. Let $\Delta \in \mathbb{R}^n_+$. Then

$$\begin{aligned}
&P(\mathbf{X} \not\leq \Delta) = P(\cup_{i=1}^{n}\{|x^i| > \Delta^i\}) \leq \sum_{i=1}^{n} P(|x^i| > \Delta^i) \\
&= \sum_{i=1}^{n} \int_{|x^i| > \Delta^i} \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{\Sigma})}} \exp\left\{-\frac{1}{2}\mathbf{x}^T \mathbf{\Sigma}^{-1} \mathbf{x}\right\} \mathbf{dx} \\
&\leq \sum_{i=1}^{n} \int_{|x^i| > \Delta^i} \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{\Sigma})}} \exp\left\{-\frac{1}{2}\lambda_{\min}(\mathbf{\Sigma}^{-1})\mathbf{x}^T \mathbf{x}\right\} \mathbf{dx} \\
&= \frac{1}{\sqrt{2\pi \det(\mathbf{\Sigma}) \lambda^{n-1}_{\min}(\mathbf{\Sigma}^{-1})}} \sum_{i=1}^{n} 2 \int_{\Delta^i}^{\infty} \exp\left\{-\frac{1}{2}\lambda_{\min}(\mathbf{\Sigma}^{-1})(x^i)^2\right\} dx^i \\
&\leq \frac{2}{\sqrt{2\pi \det(\mathbf{\Sigma}) \lambda^{n-1}_{\min}(\mathbf{\Sigma}^{-1})}} \sum_{i=1}^{n} \int_{\Delta^i}^{\infty} \frac{x^i}{\Delta_i} \exp\left\{-\frac{1}{2}\lambda_{\min}(\mathbf{\Sigma}^{-1})(x^i)^2\right\} dx^i \\
&= \frac{2}{\sqrt{2\pi \det(\mathbf{\Sigma}) \lambda^{n-1}_{\min}(\mathbf{\Sigma}^{-1})}} \sum_{i=1}^{n} \frac{1}{\Delta^i} \left[-\frac{\exp\left\{-\frac{1}{2}\lambda_{\min}(\mathbf{\Sigma}^{-1})(x^i)^2\right\}}{\lambda_{\min}(\mathbf{\Sigma}^{-1})}\right]_{\Delta^i}^{\infty} \\
&= C \sum_{i=1}^{n} \frac{1}{\Delta^i} \exp\left\{-\frac{1}{2}\lambda_{\min}(\mathbf{\Sigma}^{-1})(\Delta^i)^2\right\} \leq C \sum_{i=1}^{n} \exp\left\{-\frac{1}{2}\lambda_{\min}(\mathbf{\Sigma}^{-1})(\Delta^i)^2\right\},
\end{aligned}$$

where the last line follows when $\Delta^i \geq 1$ for all $1 \leq i \leq n$. We will see later that we can ensure this condition is met in our application of the above bound. We have also defined the constant $C = 2/(\sqrt{2\pi \det(\mathbf{\Sigma})\lambda_{\min}^{n+1}(\mathbf{\Sigma}^{-1})})$. The definition of an eigenvalue shows that eigenvalues of $\mathbf{\Sigma}^{-1}$ are the inverse eigenvalues of $\mathbf{\Sigma}$ and thus $\lambda_{\min}(\mathbf{\Sigma}^{-1}) = 1/\lambda_{\max}(\mathbf{\Sigma})$. This gives the desired bound. □

Proof of Theorem 2.4.5: *i)* **Exponential Bound.** Note that

$$P(\tau_{z+1} - \tau_z > k \mid \mathbf{x}_{\tau_z}, \Delta_{\tau_z}) = P\left(\bigcap_{s=1}^{k} \left\{ |\mathbf{x}_{\tau_z+s} + \mathbf{v}_{\tau_z+s}| \nleq \frac{K}{2}\Delta_{\tau_z+s} \right\} \middle| \mathbf{x}_{\tau_z}, \Delta_{\tau_z} \right)$$
$$= P\left(|\mathbf{x}_{\tau_z+k} + \mathbf{v}_{\tau_z+k}| \nleq \frac{K}{2}\Delta_{\tau_z+k} \middle| \bigcap_{s=1}^{k-1} \left\{ |\mathbf{x}_{\tau_z+s} + \mathbf{v}_{\tau_z+s}| \nleq \frac{K}{2}\Delta_{\tau_z+s} \right\}, \mathbf{x}_{\tau_z}, \Delta_{\tau_z} \right)$$
$$P\left(\bigcap_{s=1}^{k-1} \left\{ |\mathbf{x}_{\tau_z+s} + \mathbf{v}_{\tau_z+s}| \nleq \frac{K}{2}\Delta_{\tau_z+s} \right\} \middle| \mathbf{x}_{\tau_z}, \Delta_{\tau_z} \right)$$
$$\leq P\left(|\mathbf{x}_{\tau_z+k} + \mathbf{v}_{\tau_z+k}| \nleq \frac{K}{2}\Delta_{\tau_z+k} \middle| \bigcap_{s=1}^{k-1} \left\{ |\mathbf{x}_{\tau_z+s} + \mathbf{v}_{\tau_z+s}| \nleq \frac{K}{2}\Delta_{\tau_z+s} \right\}, \mathbf{x}_{\tau_z}, \Delta_{\tau_z} \right)$$
$$= P\left(|\mathbf{x}_{\tau_z+k} + \mathbf{v}_{\tau_z+k}| \nleq \frac{K}{2}\Delta_{\tau_z+k} \middle| \tau_{z+1} - \tau_z > k-1, \mathbf{x}_{\tau_z}, \Delta_{\tau_z} \right). \tag{2.9}$$

Let $\lambda \in \mathbb{R}$. The case of $\lambda \in \mathbb{C}$ is similar and we omit it. Let us define the noise vector $\mathbf{w}_{\tau_z,k} = \frac{\mathbf{A}^k}{\lambda^k}(-\mathbf{v}_{\tau_z} + \sum_{s=0}^{k-1} \mathbf{A}^{-1-s}\mathbf{w}_{\tau_z+s}) + \frac{\mathbf{v}_{\tau_z+k}}{\lambda^k}$ and note that it is multivariate Gaussian. Before obtaining our bound, we define $\xi = \lceil |\lambda| + \epsilon \rceil / (|\lambda| + \epsilon - \eta) > 1$. We let N denote the nilpotent matrix with ones on the upper diagonal and all other entries zero of appropriate size. Note that $N^s = 0$ for all $s \geq n$. Under our control policy, as described in Section 2.3, we know that $|(\mathbf{x}_{\tau_z} + \mathbf{v}_{\tau_z}) - \hat{\mathbf{x}}_{\tau_z}| \leq \frac{1}{2}\Delta_{\tau_z}$. It then follows that

$$P\left(|\mathbf{x}_{\tau_z+k} + \mathbf{v}_{\tau_z+k}| \nleq \frac{K}{2}\Delta_{\tau_z+k} \middle| \tau_{z+1} - \tau_z > k-1, \mathbf{x}_{\tau_z}, \Delta_{\tau_z} \right)$$
$$= P\Bigg(\left| \mathbf{A}^k \mathbf{x}_{\tau_z} + \mathbf{A}^{k-1}\mathbf{u}_{\tau_z} + \sum_{s=0}^{k-1} \mathbf{A}^{k-1-s}\mathbf{w}_{\tau_z+s} + \mathbf{v}_{\tau_z+k} \right|$$
$$\nleq \frac{K}{2}\Delta_{\tau_z+k} \Bigg| \tau_{z+1} - \tau_z > k-1, \mathbf{x}_{\tau_z}, \Delta_{\tau_z} \Bigg)$$
$$= P\Bigg(\left| \mathbf{A}^k (\mathbf{x}_{\tau_z} - \hat{\mathbf{x}}_{\tau_z} + \mathbf{v}_{\tau_z} - \mathbf{v}_{\tau_z}) + \sum_{s=0}^{k-1} \mathbf{A}^{k-1-s}\mathbf{w}_{\tau_z+s} + \mathbf{v}_{\tau_z+k} \right|$$

$$
\begin{aligned}
&\qquad \not\leq \frac{K}{2}\Delta_{\tau_z+k} \Bigg| \tau_{z+1}-\tau_z > k-1, \mathbf{x}_{\tau_z}, \Delta_{\tau_z}\Bigg) \\
&= P\Bigg(\Bigg|(\lambda I+N)^k(\mathbf{x}_{\tau_z}+\mathbf{v}_{\tau_z}-\hat{\mathbf{x}}_{\tau_z})+\mathbf{A}^k\left(-\mathbf{v}_{\tau_z}+\sum_{s=0}^{k-1}\mathbf{A}^{-1-s}\mathbf{w}_{\tau_z+s}\right)+\mathbf{v}_{\tau_z+k}\Bigg| \\
&\qquad \not\leq \frac{K}{2}\Delta_{\tau_z+k} \Bigg| \tau_{z+1}-\tau_z > k-1, \mathbf{x}_{\tau_z}, \Delta_{\tau_z}\Bigg) \\
&\leq P\Bigg(\Bigg|\left\{\lambda^k+\sum_{s=1}^{k}\binom{k}{s}\lambda^{k-s}N^s\right\}(\mathbf{x}_{\tau_z}+\mathbf{v}_{\tau_z}-\hat{\mathbf{x}}_{\tau_z})+\lambda^k\mathbf{w}_{\tau_z,k}\Bigg| \\
&\qquad \not\leq \frac{K}{2}\rho^{k-1}|\lambda|^{k-1}\frac{|\lambda|}{|\lambda|+\epsilon-\eta}\Delta_{\tau_z}\Bigg|\mathbf{x}_{\tau_z},\Delta_{\tau_z}\Bigg) \\
&\leq P\Bigg(\left\{|\lambda|^k+\sum_{s=1}^{n-1}\binom{k}{s}|\lambda|^{k-s}N^s\right\}|\mathbf{x}_{\tau_z}+\mathbf{v}_{\tau_z}-\hat{\mathbf{x}}_{\tau_z}|+|\lambda|^k|\mathbf{w}_{\tau_z,k}| \\
&\qquad \not\leq \rho^{k-1}|\lambda|^k\xi\frac{1}{2}\Delta_{\tau_z}\Bigg|\mathbf{x}_{\tau_z},\Delta_{\tau_z}\Bigg) \\
&\leq P\Bigg(|\lambda|^k\left\{\frac{1}{2}\Delta_{\tau_z}+\delta\frac{1}{2}\Delta_{\tau_z}\sum_{s=1}^{n-1}\binom{k}{s}|\lambda|^{-s}+|\mathbf{w}_{\tau_z,k}|\right\}\not\leq\rho^{k-1}|\lambda|^k\xi\frac{1}{2}\Delta_{\tau_z}\Bigg|\mathbf{x}_{\tau_z},\Delta_{\tau_z}\Bigg) \qquad (2.10) \\
&\leq P\Bigg(\frac{1}{2}\Delta_{\tau_z}+\delta\frac{1}{2}\Delta_{\tau_z}nk^n+|\mathbf{w}_{\tau_z,k}|\not\leq\rho^{k-1}\xi\frac{1}{2}\Delta_{\tau_z}\Bigg|\mathbf{x}_{\tau_z},\Delta_{\tau_z}\Bigg) \\
&\leq P\Bigg(|\mathbf{w}_{\tau_z,k}|\not\leq(\rho^{k-1}\xi-1-\delta nk^n)\frac{1}{2}\Delta_{\tau_z}\Bigg|\mathbf{x}_{\tau_z},\Delta_{\tau_z}\Bigg) \qquad (2.11) \\
&\leq P\Bigg(|\mathbf{w}_{\tau_z,k}|\not\leq(\rho')^{k-1}\frac{1}{2}\Delta_{\tau_z}\Bigg|\mathbf{x}_{\tau_z},\Delta_{\tau_z}\Bigg)\leq 2\sqrt{\frac{\lambda_{\max}^{n+1}(\mathbf{\Sigma}_{\tau_z,k})}{2\pi\det(\mathbf{\Sigma}_{\tau_z,k})}}\sum_{i=1}^{n}\exp\left\{-\frac{(\rho')^{2(k-1)}(\Delta_{\tau_z}^i)^2}{8\lambda_{\max}(\mathbf{\Sigma}_{\tau_z,k})}\right\}, \qquad (2.12)
\end{aligned}
$$

where (2.10) follows from our bin ordering. Equations (2.11) and (2.12) hold for all $k \geq H$ for some H sufficiently large and in the special case of $k = 1$. In the case $k = 1$ we choose δ sufficiently small such that $\xi - 1 - \delta n > 0$. Equation (2.12) holds for some $1 < \rho' < \rho$ since we need only show that $\rho^{k-1}\xi - 1 - \delta nk^n > (\rho')^{k-1}$ for sufficiently large k and this follows since $\lim_{k\to\infty} \rho^{k-1}/(\rho')^{k-1} = \infty$ and $\lim_{k\to\infty}(-1-\delta nk^n)/(\rho')^{k-1} = 0$

by L'Hôpital's rule. In (2.12), we have used Lemma 2.5.2 with the zero mean Gaussian vector $\mathbf{w}_{\tau_z,k}$ and denoted its covariance matrix by $\mathbf{\Sigma}_{\tau_z,k}$. From (2.11) with $k=1$, we can see that (b) of Theorem 2.4.5 holds.

In order to bound (2.12) further, we define the covariance matrices $\Sigma_{\mathbf{v}} = E[\mathbf{v}_s\mathbf{v}_s^T]$, $\Sigma_{\mathbf{v},\mathbf{w}} = E[\mathbf{v}_s\mathbf{w}_s^T]$ and $\Sigma_{\mathbf{w}} = E[\mathbf{w}_s\mathbf{w}_s^T]$. Then

$$
\begin{aligned}
\Sigma_{\tau_z,k} = E[\mathbf{w}_{\tau_z,k}\mathbf{w}_{\tau_z,k}^T] = E&\left\{\frac{\mathbf{A}^k}{\lambda^k}\left(-\mathbf{v}_{\tau_z} + \sum_{s=0}^{k-1}\mathbf{A}^{-1-s}\mathbf{w}_{\tau_z+s}\right) + \frac{\mathbf{v}_{\tau_z+k}}{\lambda^k}\right\} \\
&\left\{\frac{\mathbf{A}^k}{\lambda^k}\left(-\mathbf{v}_{\tau_z} + \sum_{s=0}^{k-1}\mathbf{A}^{-1-s}\mathbf{w}_{\tau_z+s}\right) + \frac{\mathbf{v}_{\tau_z+k}}{\lambda^k}\right\}^T \\
= \frac{\mathbf{A}^k}{\lambda^k}&\left\{\Sigma_{\mathbf{v}} - \Sigma_{\mathbf{v},\mathbf{w}}(\mathbf{A}^{-1})^T - \mathbf{A}^{-1}\Sigma_{\mathbf{v},\mathbf{w}}^T + \sum_{s=0}^{k-1}\mathbf{A}^{-1-s}\Sigma_{\mathbf{w}}(\mathbf{A}^{-1-s})^T\right\}\frac{(\mathbf{A}^k)^T}{\lambda^k} + \frac{\Sigma_{\mathbf{v}}}{\lambda^{2k}},
\end{aligned}
$$

where we have used the independence of $\{\mathbf{w}_s\}$, $\{\mathbf{v}_s\}$ across time and the independence of $\mathbf{v}_{s_1}$ and $\mathbf{w}_{s_2}$ for $s_1 \neq s_2$. Since both processes are zero mean, the cross terms are zero.

Recall that

$$
\mathbf{A}^k = \begin{bmatrix}
\lambda^k & \binom{k}{1}\lambda^{k-1} & \cdots & \binom{k}{n-1}\lambda^{k-n+1} \\
 & \lambda^k & \ddots & \vdots \\
 & 0 & \ddots & \binom{k}{1}\lambda^{k-1} \\
 & & & \lambda^k
\end{bmatrix}
$$

and let $Tr(\cdot)$ denote the trace of its argument. We get that

$$
Tr(\mathbf{A}^k(\mathbf{A}^k)^T) = \sum_{\ell=0}^{n-1}\sum_{s=0}^{\ell}\binom{k}{s}^2\lambda^{2(k-s)} \leq n\sum_{s=0}^{n-1}\binom{k}{s}^2\lambda^{2(k-s)} \leq n\lambda^{2k}\sum_{s=0}^{n-1}\binom{k}{s}^2 \leq \lambda^{2k}n^2k^{2n}.
$$

Similarly, we can see that $Tr(\mathbf{A}^{k-1-s}(\mathbf{A}^{k-1-s})^T) \leq \lambda^{2k}n^2k^{2n}$ for all $0 \leq s \leq k-1$.

Define

$$
\begin{aligned}
\mathbf{\Sigma}_1 &= E[(-\mathbf{v}_{\tau_z} + \mathbf{A}^{-1}\mathbf{w}_{\tau_z})(-\mathbf{v}_{\tau_z} + \mathbf{A}^{-1}\mathbf{w}_{\tau_z})^T] \\
&= \mathbf{\Sigma}_{\mathbf{v}} - \mathbf{\Sigma}_{\mathbf{v},\mathbf{w}}(\mathbf{A}^{-1})^T - \mathbf{A}^{-1}\mathbf{\Sigma}_{\mathbf{v},\mathbf{w}}^T + \mathbf{A}^{-1}\mathbf{\Sigma}_{\mathbf{w}}(\mathbf{A}^{-1})^T.
\end{aligned}
$$

For symmetric matrices, there exists a basis of eigenvectors. Thus, for $\mathbf{\Sigma}_{\tau_z,k}$, there exists a vector of unit length $\mathbf{e} \in \mathbb{R}^n$ such that

$$\begin{aligned}
\lambda_{\max}(\mathbf{\Sigma}_{\tau_z,k}) = \mathbf{e}^T\mathbf{\Sigma}_{\tau_z,k}\mathbf{e} = \frac{1}{\lambda^{2k}}(\mathbf{e}^T\mathbf{A}^k)\mathbf{\Sigma}_1((\mathbf{A}^k)^T\mathbf{e}) \\
+ \frac{1}{\lambda^{2k}}\sum_{s=1}^{k-1}(\mathbf{e}^T\mathbf{A}^{k-1-s})\mathbf{\Sigma}_{\mathbf{w}}((\mathbf{A}^{k-1-s})^T\mathbf{e}) + \frac{1}{\lambda^{2k}}\mathbf{e}^T\mathbf{\Sigma}_{\mathbf{v}}\mathbf{e} \\
\leq \frac{1}{\lambda^{2k}}\lambda_{\max}(\mathbf{\Sigma}_1)\mathbf{e}^T\mathbf{A}^k(\mathbf{A}^k)^T\mathbf{e} \\
+ \frac{1}{\lambda^{2k}}\lambda_{\max}(\mathbf{\Sigma}_{\mathbf{w}})\sum_{s=1}^{k-1}\mathbf{e}^T\mathbf{A}^{k-1-s}(\mathbf{A}^{k-1-s})^T\mathbf{e} + \frac{1}{\lambda^{2k}}\lambda_{\max}(\mathbf{\Sigma}_{\mathbf{v}})\mathbf{e}^T\mathbf{e} \\
\leq \frac{1}{\lambda^{2k}}\lambda_{\max}(\mathbf{\Sigma}_1)\lambda_{\max}(\mathbf{A}^k(\mathbf{A}^k)^T)\mathbf{e}^T\mathbf{e} \\
+ \frac{1}{\lambda^{2k}}\lambda_{\max}(\mathbf{\Sigma}_{\mathbf{w}})\lambda_{\max}(\mathbf{A}^{k-1-s}(\mathbf{A}^{k-1-s})^T)\sum_{s=1}^{k-1}\mathbf{e}^T\mathbf{e} + \lambda_{\max}(\mathbf{\Sigma}_{\mathbf{v}}) \\
\leq \frac{1}{\lambda^{2k}}\lambda_{\max}(\mathbf{\Sigma}_1)Tr(\mathbf{A}^k(\mathbf{A}^k)^T) + k\frac{1}{\lambda^{2k}}\lambda_{\max}(\mathbf{\Sigma}_{\mathbf{w}})Tr(\mathbf{A}^{k-1-s}(\mathbf{A}^{k-1-s})^T) + \lambda_{\max}(\mathbf{\Sigma}_{\mathbf{v}}) \\
\leq n^2k^{2n+1}(\lambda_{\max}(\mathbf{\Sigma}_1) + \lambda_{\max}(\mathbf{\Sigma}_{\mathbf{w}}) + \lambda_{\max}(\mathbf{\Sigma}_{\mathbf{v}})).
\end{aligned}$$

Note that for any matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ and any vector $\mathbf{x} \in \mathbb{R}^n$ that

$$\mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x} = (\mathbf{A}\mathbf{x})^T\mathbf{A}\mathbf{x} > 0$$

and thus $\mathbf{A}^T\mathbf{A}$ is positive definite and has all eigenvalues positive. Thus, in the above we are justified in claiming that $\lambda_{\max}(\mathbf{A}^k(\mathbf{A}^k)^T) \leq Tr(\mathbf{A}^k(\mathbf{A}^k)^T)$.

Recall Minkowski's Determinant Theorem (see for example [25]). For nonnegative definite $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$, it follows that $\det(\mathbf{A}+\mathbf{B}) \geq \det(\mathbf{A}) + \det(\mathbf{B})$.

Using the above bound and the identity $\det(\mathbf{A}) = \det\mathbf{A}^T$ we get

$$\det(\mathbf{\Sigma}_{\tau_z,k}) \geq \frac{1}{\lambda^{2nk}}\det(\mathbf{A}^k)^2\det\left(\mathbf{\Sigma}_1 + \sum_{s=1}^{k-1}\mathbf{A}^{-1-s}\mathbf{\Sigma}_{\mathbf{w}}(\mathbf{A}^{-1-s})^T\right) + \frac{1}{\lambda^{2nk}}\det(\mathbf{\Sigma}_{\mathbf{v}})$$
$$\geq \det(\mathbf{\Sigma}_1) + \det\left(\sum_{s=1}^{k-1}\mathbf{A}^{-1-s}\mathbf{\Sigma}_{\mathbf{w}}(\mathbf{A}^{-1-s})^T\right) \geq \det(\mathbf{\Sigma}_1) + \sum_{s=1}^{k-1}\det(\mathbf{A}^{-1-s}\mathbf{\Sigma}_{\mathbf{w}}(\mathbf{A}^{-1-s})^T)$$

$$= \det(\mathbf{\Sigma}_1) + \det(\mathbf{\Sigma}_{\mathbf{w}}) \sum_{s=1}^{k-1} (\lambda^{-1-s})^{2n} \geq \det(\mathbf{\Sigma}_1).$$

Defining the constants $c_1 = n^2(\lambda_{\max}(\mathbf{\Sigma}_1) + \lambda_{\max}(\mathbf{\Sigma}_{\mathbf{w}}) + \lambda_{\max}(\mathbf{\Sigma}_{\mathbf{v}}))$ and $c_2 = \det(\mathbf{\Sigma}_1)$ we have obtained the bounds

$$\lambda_{\max}(\mathbf{\Sigma}_{\tau_z,k}) \leq c_1 k^{2n+1}, \quad \det(\mathbf{\Sigma}_{\tau_z,k}) \geq c_2. \tag{2.13}$$

Combining (2.9), (2.12) and (2.13) we get that

$$\begin{aligned} P(\tau_{z+1} - \tau_z > k \mid \mathbf{x}_{\tau_z}, \Delta_{\tau_z}) &\leq 2\sqrt{\frac{(c_1 k^{2n+1})^{n+1}}{2\pi c_2}} \sum_{i=1}^{n} \exp\left\{ -\frac{(\rho')^{2(k-1)}(\Delta^i_{\tau_z})^2}{8c_1 k^{2n+1}} \right\} \\ &\leq C k^{n^2+\frac{3}{2}n+\frac{1}{2}} \exp\left\{ -\frac{(\rho')^{2(k-1)}}{8c_1 k^{2n+1}} \right\}, \end{aligned} \tag{2.14}$$

for $k > H$, where C is the appropriate constant. Note that the last line follows since $\Delta^i_{\tau_z} > 1$ and thus our bound holds for any initial condition Δ_{τ_z}.

ii) **Geometric Bound.** Note that in (2.14) we have a double exponential in k since $(\rho')^{2(k-1)} = e^{(k-1)2\log(\rho')}$. Let $a, b, c > 0$ and recall that $\lim_{k\to\infty} e^k/((a+b)k^{c+1}) = \infty$ by L'Hôpital's rule. This means that for all $L > 0$ there exists an N such that $e^k/((a+b)k^{c+1}) > L$, for all $k \geq N$. Thus, $e^k/k^c > L(a+b)k$, for all $k \geq N$. Then choosing N large enough so that $(a+b)N > 1$ and subtracting $(a+b)k$ we get $e^k/k^c - (a+b)k > (L-1)(a+b)k > L-1$, for all $k \geq N$. Therefore, we have that $\lim_{k\to\infty} e^k/k^c - (a+b)k = \infty$. Since $\log(k) \leq k$ for $k \geq 1$, comparing with the above we find that $\lim_{k\to\infty} e^k/k^c - a\log(k) - bk = \infty$.

Let $Q(k)$ be a polynomial of finite degree m. We can write $Q(k) = a_0 + a_1 k + \cdots + a_m k^m$, for some coefficients $a_0, \ldots, a_m \in \mathbb{R}$. Let $r > 1$ and consider

$$\begin{aligned} \lim_{k\to\infty} r^k Q(k) \exp\left\{ -\frac{e^k}{k^c} \right\} &= \lim_{k\to\infty} \sum_{i=0}^{m} a_i k^i r^k \exp\left\{ -\frac{e^k}{k^c} \right\} \\ &= \lim_{k\to\infty} \sum_{i=0}^{m} a_i \exp\left\{ -\frac{e^k}{k^c} + i\log(k) + \log(r)k \right\} = 0. \end{aligned} \tag{2.15}$$

Combining (2.14) and (2.15) gives the result. □

Proof of Lemma 2.4.6: We take $\lambda \in \mathbb{R}$. The proof for $\lambda \in \mathbb{C}$ is identical. We use the abbreviations:

$$\begin{aligned} P_z(k) &= P(\tau_{z+1} - \tau_z = k \mid \mathbf{x}_{\tau_z}, \Delta_{\tau_z}), \\ P_z(k \mid Y) &= P(\tau_{z+1} - \tau_z = k \mid Y, \mathbf{x}_{\tau_z}, \Delta_{\tau_z}), \\ \bar{P}_z(k) &= P(\tau_{z+1} - \tau_z > k \mid \mathbf{x}_{\tau_z}, \Delta_{\tau_z}), \\ \bar{P}_z(k \mid Y) &= P(\tau_{z+1} - \tau_z > k \mid Y, \mathbf{x}_{\tau_z}, \Delta_{\tau_z}). \end{aligned}$$

Put $r > \rho^2|\lambda|^2$. Note that $P(\tau_{z+1} - \tau_z = k \mid \mathbf{x}_{\tau_z}, \Delta_{\tau_z}) \leq P(\tau_{z+1} - \tau_z > k-1 \mid \mathbf{x}_{\tau_z}, \Delta_{\tau_z})$. Then using Theorem 2.4.5, we can bound the first term in (2.7) using the law of iterated expectations as follows

$$\begin{aligned} &E\left[\sum_{s=\tau_z}^{\tau_{z+1}-1} (\Delta_s^1)^2 \middle| \mathbf{x}_{\tau_z}, \Delta_{\tau_z}\right] = \sum_{k=1}^{\infty} P_z(k) \sum_{s=0}^{k-1} E[(\Delta_{\tau_z+s}^1)^2 | \tau_{z+1} - \tau_z = k, \Delta_{\tau_z}^1] \\ &\leq \sum_{k=1}^{\infty} P_z(k) \sum_{s=0}^{k-1} \rho^{2s}|\lambda|^{2s}(\Delta_{\tau_z}^1)^2 \leq (\Delta_{\tau_z}^1)^2 \sum_{k=1}^{\infty} k\bar{P}_z(k-1)\rho^{2k}|\lambda|^{2k} \\ &\leq (\Delta_{\tau_z}^1)^2 \left(\sum_{k=1}^{H} k\bar{P}_z(k-1)\rho^{2k}|\lambda|^{2k} + \sum_{k=H+1}^{\infty} r\left(\frac{\rho^2|\lambda|^2}{r}\right)^k\right) = (\Delta_{\tau_z}^1)^2 G_1. \end{aligned} \tag{2.16}$$

We have defined

$$G_1 = \sum_{k=1}^{H} k\bar{P}_z(k-1)|\lambda|^{2k} + r \sum_{k=H+1}^{\infty} (\rho^2|\lambda|^2/r)^k < \infty$$

and used

$$kP_z(k) \leq k\bar{P}_z(k-1) \leq r^{-(k-1)}$$

for $k > H$. The series on the right converges since it is geometric. Similarly, we can bound the term $E[(\Delta_{\tau_{z+1}}^1)^2 \mid \mathbf{x}_{\tau_z}, \Delta_{\tau_z}]$. Using the law of iterated expectations, we get

$$\begin{aligned} &E[(\Delta_{\tau_{z+1}}^1)^2 \mid \mathbf{x}_{\tau_z}, \Delta_{\tau_z}] = P_z(1)E[(\Delta_{\tau_{z+1}}^1)^2 \mid \tau_{z+1} - \tau_z = 1, \Delta_{\tau_z}] \\ &\quad + \bar{P}_z(1)E[(\Delta_{\tau_{z+1}}^1)^2 \mid \tau_{z+1} - \tau_z > 1, \mathbf{x}_{\tau_z}, \Delta_{\tau_z}] = P_z(1)E[(\Delta_{\tau_{z+1}}^1)^2 \mid \tau_{z+1} - \tau_z = 1, \Delta_{\tau_z}] \\ &\quad + \bar{P}_z(1)E[E[(\Delta_{\tau_{z+1}}^1)^2 \mid \tau_{z+1} - \tau_z > 1, \tau_{z+1} - \tau_z, \mathbf{x}_{\tau_z}, \Delta_{\tau_z}] \mid \tau_{z+1} - \tau_z > 1, \mathbf{x}_{\tau_z}, \Delta_{\tau_z}] \end{aligned}$$

$$\leq P_z(1)E[(\Delta^1_{\tau_{z+1}})^2 \mid \tau_{z+1}-\tau_z=1,\Delta_{\tau_z}] + \bar{P}_z(1)\sum_{k=2}^{\infty} P_z(k \mid \tau_{z+1}-\tau_z>1)\rho^{2k}|\lambda|^{2k}(\Delta^1_{\tau_z})^2$$
$$= P_z(1)E[(\Delta^1_{\tau_{z+1}})^2 \mid \tau_{z+1}-\tau_z=1,\Delta_{\tau_z}] + \bar{P}_z(1)G_2(\Delta^1_{\tau_z})^2, \tag{2.17}$$

where we have defined

$$G_2 = \sum_{k=2}^{\infty} P_z(k \mid \tau_{z+1}-\tau_z>1)\rho^{2k}|\lambda|^{2k} < \infty.$$

Convergence comes from the geometric decay, as in the previous bound. Note that the geometric bound in Theorem 2.4.5 still holds with $P_z(k \mid \tau_{z+1}-\tau_z>1)$ in place of $P_z(k)$ since we obtain our bound by looking only at the τ_z+k term, as can be seen in (2.9).

There exists a ζ such that $0<\zeta<1-(|\lambda|/(|\lambda|+\epsilon-\eta))^2$. We know from Theorem 2.4.5 that $\lim_{\Delta_{\tau_z}\to\infty}\bar{P}_z(1)=0$. Recall that $\Delta^i_s \geq \bar{L}^i$ for all $t\in\mathbb{N}$. Then, we choose $\mathbf{L}$ large enough to get an appropriate $\bar{\mathbf{L}}$ such that $\bar{P}_z(1)G_2<\zeta$. We put

$$\gamma = \frac{1-\left(\frac{|\lambda|}{|\lambda|+\epsilon-\eta}\right)^2-\zeta}{G_1},$$

so that $\gamma>0$. Note that since our bound in Theorem 2.4.5 (a) does not depend on Δ_{τ_z} and $\bar{P}_z(1)G_2<\zeta$ holds for all Δ_{τ_z} by our choice of $\bar{\mathbf{L}}$, it follows that G_1, ζ and hence γ have no dependence on Δ_{τ_z}, as required. Now, if $\Delta_{\tau_z}\notin S_\Delta$ then we have that $\Delta^1_{\tau_z}\geq F$ since $\Delta^1_s\geq\Delta^2_s\geq\cdots\geq\Delta^n_s$ for all $t\in\mathbb{N}$ by construction. Since $F>L^1$, the bin size shrinks and

$$E[(\Delta^1_{\tau_{z+1}})^2 \mid \tau_{z+1}-\tau_z=1,\Delta_{\tau_z}] = \left(\frac{|\lambda|}{|\lambda|+\epsilon-\eta}\right)^2(\Delta^1_{\tau_z})^2.$$

If $\Delta_{\tau_z}\in S_\Delta$ then we use the simple bound $E[(\Delta^1_{\tau_{z+1}})^2 \mid \tau_{z+1}-\tau_z=1,\Delta_{\tau_z}] \leq \rho^2|\lambda|^2(\Delta^1_{\tau_z})^2$.

From the above, we have the following bounds. If $\Delta_{\tau_z}\notin S_\Delta$ then

$$E[(\Delta^1_{\tau_{z+1}})^2 \mid \mathbf{x}_{\tau_z},\Delta_{\tau_z}] \leq (\Delta^1_{\tau_z})^2\left\{\left(\frac{|\lambda|}{|\lambda|+\epsilon-\eta}\right)^2+\zeta\right\}. \tag{2.18}$$

If $\Delta_{\tau_z} \in S_\Delta$ then

$$E[(\Delta^1_{\tau_{z+1}})^2 \mid \mathbf{x}_{\tau_z}, \Delta_{\tau_z}] \leq (\Delta^1_{\tau_z})^2\{\rho^2|\lambda|^2 + \zeta\}. \tag{2.19}$$

In the case $\Delta_{\tau_z} \notin S_\Delta$ we apply (2.16), (2.17) and (2.18) to get

$$\begin{aligned}
&\gamma E\left[\sum_{s=\tau_z}^{\tau_{z+1}-1} (\Delta^1_s)^2 \middle| \mathbf{x}_{\tau_z}, \Delta_{\tau_z}\right] \leq (\Delta^1_{\tau_z})^2 \gamma G_1 \\
&= (\Delta^1_{\tau_z})^2 \left\{1 - \left(\frac{|\lambda|}{|\lambda| + \epsilon - \eta}\right)^2 - \zeta\right\} \leq (\Delta^1_{\tau_z})^2 - E[(\Delta^1_{\tau_{z+1}})^2 | \mathbf{x}_{\tau_z}, \Delta_{\tau_z}].
\end{aligned}$$

In the case $\Delta_{\tau_z} \in S_\Delta$ we apply (2.16), (2.17) and (2.19) to get

$$\begin{aligned}
&\gamma E\left[\sum_{s=\tau_z}^{\tau_{z+1}-1} (\Delta^1_s)^2 \middle| \mathbf{x}_{\tau_z}, \Delta_{\tau_z}\right] \leq (\Delta^1_{\tau_z})^2 \gamma G_1 = (\Delta^1_{\tau_z})^2 \left\{1 - \left(\frac{|\lambda|}{|\lambda| + \epsilon - \eta}\right)^2 - \zeta\right\} \\
&= (\Delta^1_{\tau_z})^2 - (\Delta^1_{\tau_z})^2\{\rho^2|\lambda|^2 + \zeta\} + (\Delta^1_{\tau_z})^2 \left\{\rho^2|\lambda|^2 - \left(\frac{|\lambda|}{|\lambda| + \epsilon - \eta}\right)^2\right\} \\
&\leq (\Delta^1_{\tau_z})^2 - E[(\Delta^1_{\tau_{z+1}})^2 \mid \mathbf{x}_{\tau_z}, \Delta_{\tau_z}] + F^2 \left\{\rho^2|\lambda|^2 - \left(\frac{|\lambda|}{|\lambda| + \epsilon - \eta}\right)^2\right\}.
\end{aligned}$$

We set $b = F^2\{\rho^2|\lambda|^2 - (|\lambda|/(|\lambda| + \epsilon - \eta))^2\}$. Since $\Delta_{\tau_z} \in S_\Delta$ if and only if $(\mathbf{x}_{\tau_z}, \Delta_{\tau_z}) \in S$, we obtain Lemma 2.4.6. □

Proof of Theorem 2.4.7: Let $\lambda \in \mathbb{R}$ and let $x^{n+1}_s = 0$. For $\lambda \in \mathbb{C}$, the proof is similar and we omit it. Using the law of total expectation we get

$$\begin{aligned}
&E\left[\sum_{s=\tau_z}^{\tau_{z+1}-1} (x^i_s)^2 \middle| \mathbf{x}_{\tau_z}, \Delta_{\tau_z}\right] = E\Big[E\Big[(x^i_{\tau_z})^2 + \sum_{s=\tau_z+1}^{\tau_{z+1}-1} (\lambda x^i_{s-1} + x^{i+1}_{s-1} + u^i_{s-1} + w^i_{s-1})^2 \\
&\qquad \Big| \tau_{z+1} - \tau_z, \mathbf{x}_{\tau_z}, \Delta_{\tau_z}\Big]\Big| \mathbf{x}_{\tau_z}, \Delta_{\tau_z}\Big] \\
&= \sum_{k=1}^{\infty} P_z(k) E\left[(x^i_{\tau_z})^2 + \sum_{s=1}^{k-1}\left(\lambda^s x^i_{\tau_z} + \lambda^{s-1} x^{i+1}_{\tau_z} - \lambda^{s-1} u^i_{\tau_z} + \sum_{j=1}^{s-1} \lambda^{s-1-j} x^{i+1}_{\tau_z+j}\right.\right. \\
&\qquad \left.\left. + \sum_{j=0}^{s-1} \lambda^{s-1-j} w^i_{\tau_z+j}\right)^2 \middle| \mathbf{x}_{\tau_z}, \Delta_{\tau_z}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} P_z(k) E\Bigg[(x^i_{\tau_z})^2 + \sum_{s=1}^{k-1} \Bigg(\lambda^s (x^i_{\tau_z} + v^i_{\tau_z} - \hat{x}^i_{\tau_z}) + \lambda^{s-1}(x^{i+1}_{\tau_z} + v^{i+1}_{\tau_z} - \hat{x}^{i+1}_{\tau_z}) \\
&\qquad + \sum_{j=1}^{s-1} \lambda^{s-1-j} x^{i+1}_{\tau_z+j} - \lambda^s v^i_{\tau_z} - \lambda^{s-1} v^{i+1}_{\tau_z} + \sum_{j=0}^{s-1} \lambda^{s-1-j} w^i_{\tau_z+j} \Bigg)^2 \Bigg| \mathbf{x}_{\tau_z}, \Delta_{\tau_z} \Bigg] \\
&\leq \sum_{k=1}^{\infty} P_z(k) E\Bigg[(x^i_{\tau_z})^2 + 6 \sum_{s=1}^{k-1} \Big(\lambda^{2s} (x^i_{\tau_z} + v^i_{\tau_z} - \hat{x}^i_{\tau_z})^2 + \lambda^{2(s-1)} (x^{i+1}_{\tau_z} + v^{i+1}_{\tau_z} - \hat{x}^{i+1}_{\tau_z})^2 \\
&\qquad + \Bigg(\sum_{j=1}^{s-1} \lambda^{s-1-j} x^{i+1}_{\tau_z+j} \Bigg)^2 + \lambda^{2s} (v^i_{\tau_z})^2 + \lambda^{2(s-1)} (v^{i+1}_{\tau_z})^2 \\
&\qquad + \Bigg(\sum_{j=0}^{s-1} \lambda^{s-1-j} w^i_{\tau_z+j} \Bigg)^2 \Big) \Bigg| \mathbf{x}_{\tau_z}, \Delta_{\tau_z} \Bigg] \qquad (2.20) \\
&\leq 6 \sum_{k=1}^{\infty} P_z(k) E\Bigg[\frac{K^2}{4} (\Delta^i_{\tau_z})^2 + \sum_{s=1}^{k-1} \Bigg(\lambda^{2s} \left(\frac{1}{2} \Delta^i_{\tau_z} \right)^2 + \lambda^{2(s-1)} \left(\frac{1}{2} \Delta^{i+1}_{\tau_z} \right)^2 \\
&\qquad + s \sum_{j=1}^{s-1} \lambda^{2(s-1-j)} (x^{i+1}_{\tau_z+j})^2 + \lambda^{2s} (v^i_{\tau_z})^2 + \lambda^{2(s-1)} (v^{i+1}_{\tau_z})^2 \\
&\qquad + s \sum_{j=0}^{s-1} \lambda^{2(s-1-j)} (w^i_{\tau_z+j})^2 \Bigg) \Bigg| \mathbf{x}_{\tau_z}, \Delta_{\tau_z} \Bigg] \qquad (2.21) \\
&\leq 6 \sum_{k=1}^{\infty} P_z(k) \Bigg(\frac{K^2}{4} (\Delta^1_{\tau_z})^2 + \sum_{s=1}^{k-1} \Big(\lambda^{2s} \frac{1}{2} \left(\Delta^1_{\tau_z} \right)^2 + s^2 \lambda^{2s} M^{i+1} \\
&\qquad + \lambda^{2s} \sigma^2_{v,i} + \lambda^{2s} \sigma^2_{v,i+1} + s^2 \lambda^{2s} \sigma^2_{w,i} \Big) \Bigg) \\
&\leq (\Delta^1_{\tau_z})^2 6 \sum_{k=1}^{\infty} P_z(k) \left\{ \frac{K^2}{4} + \sum_{s=1}^{k-1} \lambda^{2s} \left(\frac{1}{2} + s^2 M^{i+1} + \sigma^2_{v,i} + \sigma^2_{v,i+1} + s^2 \sigma^2_{w,i} \right) \right\}. \qquad (2.22)
\end{aligned}
$$

In (2.20) and (2.21) we have used Jensen's inequality. Line (2.22) follows since we can bound $\Delta^i_s > 1$ for all $s \in \mathbb{N}$. We have defined $M^i = \sup_{s\in\mathbb{N}} E[(x^i_s)^2] < \infty$, $\sigma^2_{v,i} = E[(v^i_s)^2]$ and $\sigma^2_{w,i} = E[(w^i_s)^2]$. The fact that M^i is finite for $2 \leq i \leq n-1$ follows from induction in the proof of Theorem 2.2.1. By convention we put $M^{n+1} = 0$. Now, we apply Theorem 2.4.5 with $Q(k) = k^3$ and $r > \lambda^2$ to yield

$$\sum_{k=1}^{\infty} P_z(k) \sum_{s=1}^{k-1} \lambda^{2s} s^2 \leq \sum_{k=1}^{H} \sum_{s=1}^{k-1} P_z(k) \lambda^{2s} s^2 + \sum_{k=H+1}^{\infty} \sum_{s=1}^{k-1} \lambda^{2s} \bar{P}_z(k-1) s^2$$
$$\leq G + \sum_{k=H+1}^{\infty} \lambda^{2k} \bar{P}_z(k-1) k^3 \leq G + r \sum_{k=H+1}^{\infty} \left(\frac{\lambda^2}{r}\right)^k < \infty.$$

The last series converges since it is geometric. We have defined

$$G = \sum_{k=1}^{H} \sum_{s=1}^{k-1} P_z(k) \lambda^{2s} s^2 < \infty.$$

Therefore we can set

$$\kappa = 6 \sum_{k=1}^{\infty} P_z(k)(K^2/4 + \sum_{s=0}^{k-1} \lambda^{2s} \left(1/2 + s^2 M^{i+1} + \sigma_{v,i}^2 + \sigma_{v,i+1}^2 + s^2 \sigma_{w,i}^2\right) < \infty$$

to get the result. □

Chapter 3

Extension to a Larger Class of Noise Distributions

In Chapter 2, we considered single-station systems driven by Gaussian noise. In this chapter, we extend our results to a larger class of noise distributions.

Consider the setup from Section 2.1, except that we now allow the noise processes $\{\mathbf{w}_t\}$ and $\{\mathbf{v}_t\}$ to be i.i.d. sequences of random variables with distribution $\mathcal{Z}$, not necessarily of zero mean. That is, for any t we have $\mathbf{w}_t, \mathbf{v}_t \sim \mathcal{Z}$.

Theorem 3.0.3. *If $\mathcal{Z}$ admits a density which is positive on every open set and there exists an $\epsilon > 0$ such that $E[|\mathbf{z}|^{2+\epsilon}] < \infty$ where $\mathbf{z} \sim \mathcal{Z}$, then Theorem 2.2.1 holds with the new noise distribution. The exponent $2+\epsilon$ is applied component-wise and thus our requirement is that every component of $\mathbf{z}$ has a finite $2+\epsilon$ moment. The rate remains identical to the Gaussian case.*

Proof of Theorem 3.0.3:

We continue with (2.12) from the proof of Theorem 2.4.5. Recall that $\bar{P}_z(k) := P(\tau_{z+1} - \tau_z > k \mid \mathbf{x}_{\tau_z}, \Delta_{\tau_z})$. There exists an H such that for $k > H$ we obtain

$$
\begin{aligned}
\bar{P}_z(k) &\leq P(|\mathbf{w}_{\tau_z,k}| \nleq (\rho')^{k-1}\Delta_{\tau_z}/2 \mid \mathbf{x}_{\tau_z}, \Delta_{\tau_z}) \\
&= P(|\mathbf{w}_{\tau_z,k}|^{2+\epsilon} \nleq ((\rho')^{k-1}\Delta_{\tau_z}/2)^{2+\epsilon} \mid \mathbf{x}_{\tau_z}, \Delta_{\tau_z}) && (3.1) \\
&\leq P((\mathbf{w}_{\tau_z,k}^T \mathbf{w}_{\tau_z,k})^{\frac{2+\epsilon}{2}} > ((\rho')^{k-1}\Delta_{\tau_z}^n/2)^{2+\epsilon} \mid \mathbf{x}_{\tau_z}, \Delta_{\tau_z}) && (3.2)
\end{aligned}
$$

$$\leq \frac{E[(\mathbf{w}_{\tau_z,k}^T \mathbf{w}_{\tau_z,k})^{\frac{2+\epsilon}{2}} \mid \mathbf{x}_{\tau_z}, \Delta_{\tau_z}]}{((\rho')^{k-1}\Delta_{\tau_z}^n/2)^{2+\epsilon}} \tag{3.3}$$

$$\leq \frac{E(\mathbf{w}_{\tau_z,k}^T \mathbf{w}_{\tau_z,k})^{\frac{2+\epsilon}{2}}}{((\rho')^{2+\epsilon})^{k-1}}. \tag{3.4}$$

We apply the exponent $2+\epsilon$ component-wise in equation (3.1). We obtain (3.2) since the event being measured contains the event in the previous line. More precisely,

$$\bigcup_{i=1}^{n} \{|\mathbf{w}_{\tau_z,k}^i|^{2+\epsilon} > ((\rho')^{k-1}\Delta_{\tau_z}^i/2)^{2+\epsilon}\} \subseteq \Big\{ \Big(\sum_{i=1}^{n} (\mathbf{w}_{\tau_z,k}^i)^2 \Big)^{\frac{2+\epsilon}{2}} > ((\rho')^{k-1}\Delta_{\tau_z}^n/2)^{2+\epsilon} \Big\},$$

since we have ordered the bins $\Delta_t^1 \geq \Delta_t^2 \geq \cdots \geq \Delta_t^n \geq 1$. In (3.3) we apply Markov's inequality.

We now proceed to bound the numerator in (3.4):

$$E[(\mathbf{w}_{\tau_z,k}^T \mathbf{w}_{\tau_z,k})^{\frac{2+\epsilon}{2}}] = E[(\|\mathbf{w}_{\tau_z,k}\|_2)^{2+\epsilon}]$$

$$= E\Big[\Big(\Big\| \frac{\mathbf{A}^k}{\lambda^k}(-\mathbf{v}_{\tau_z} + \sum_{s=0}^{k-1} \mathbf{A}^{-1-s}\mathbf{w}_{\tau_z+s}) + \frac{\mathbf{v}_{\tau_z+k}}{\lambda^k} \Big\|_2\Big)^{2+\epsilon}\Big]$$

$$\leq E\Big[\Big(\Big\| \frac{\mathbf{A}^k}{\lambda^k}\mathbf{v}_{\tau_z} \Big\|_2 + \sum_{s=0}^{k-1} \Big\| \frac{\mathbf{A}^{k-1-s}}{\lambda^k}\mathbf{w}_{\tau_z+s} \Big\|_2 + \Big\| \frac{\mathbf{v}_{\tau_z+k}}{\lambda^k} \Big\|_2\Big)^{2+\epsilon}\Big] \tag{3.5}$$

$$\leq 3^{1+\epsilon} E\Big(\Big\| \frac{\mathbf{A}^k}{\lambda^k}\mathbf{v}_{\tau_z} \Big\|_2^{2+\epsilon} + k^{1+\epsilon} \sum_{s=0}^{k-1} \Big\| \frac{\mathbf{A}^{k-1-s}}{\lambda^k}\mathbf{w}_{\tau_z+s} \Big\|_2^{2+\epsilon} + \Big\| \frac{\mathbf{v}_{\tau_z+k}}{\lambda^k} \Big\|_2^{2+\epsilon} \Big) \tag{3.6}$$

$$\leq 3^{1+\epsilon} E\Big(n^{2+\frac{3}{2}\epsilon} k^{n(2+\epsilon)} \sum_{i=1}^{n} |v_{\tau_z}^i|^{2+\epsilon} + n^{2+\frac{3}{2}\epsilon} k^{2n+n\epsilon+\epsilon+1} \sum_{s=0}^{k-1} \sum_{i=1}^{n} |w_{\tau_z+s}^i|^{2+\epsilon}$$

$$+ \frac{n^{\frac{\epsilon}{2}}}{|\lambda|^{k(2+\epsilon)}} \sum_{i=1}^{n} |v_{\tau_z+k}^i|^{2+\epsilon} \Big) \tag{3.7}$$

$$\leq Ck^{(n+1)(\epsilon+2)} < \infty, \tag{3.8}$$

where $C > 0$ is some constant. The last line follows since $\{\mathbf{w}_t\}$ and $\{\mathbf{v}_t\}$ are i.i.d. sequences of random variables. In (3.5), we have applied the triangle inequality. Equation (3.6) follows from Jensen's inequality. The constant C is finite since each of the components of the noise have a finite $2+\epsilon$ moment by assumption.

There is one subtlety above worth noting. Each term in the sequences $\{\mathbf{w}_t\}$ and $\{\mathbf{v}_t\}$ is actually a finite sum of raw random variables with distribution $\mathcal{Z}$. This is described by our sampling method in Lemma 2.4.1. However, a simple application of Jensen's inequality shows that, if the raw random variables have a finite $2+\epsilon$ moment, then so do a finite linear combinition of such random variables.

The bound (3.7) follows from basic properties of symmetric, positive definite matrices and Jensen's inequality. Let us derive this bound for one of the terms explicitly.

Note that, for any matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ and any vector $\mathbf{x} \in \mathbb{R}^n$, we have

$$\mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x} = (\mathbf{A}\mathbf{x})^T\mathbf{A}\mathbf{x} > 0.$$

Thus, $(\mathbf{A}^{k-1-s})^T\mathbf{A}^{k-1-s}$ is positive definite and has all eigenvalues positive. It must be that the largest eigenvalue of this matrix is less than it's trace. More explicitly,

$$\lambda_{\max}((\mathbf{A}^{k-1-s})^T\mathbf{A}^{k-1-s}) \leq Tr((\mathbf{A}^{k-1-s})^T\mathbf{A}^{k-1-s}) \leq \lambda^{2k}n^2k^{2n}$$

where the last bound was derived in the proof of Theorem 2.4.5. It then follows that

$$\begin{aligned}
&\sum_{s=0}^{k-1}\left\|\frac{\mathbf{A}^{k-1-s}}{\lambda^k}\mathbf{w}_{\tau_z+s}\right\|_2^{2+\epsilon} \\
&= \sum_{s=0}^{k-1}\left(\frac{1}{\lambda^{2k}}\mathbf{w}_{\tau_z+s}^T(\mathbf{A}^{k-1-s})^T\mathbf{A}^{k-1-s}\mathbf{w}_{\tau_z+s}\right)^{\frac{2+\epsilon}{2}} \\
&\leq \sum_{s=0}^{k-1}\left(\frac{1}{\lambda^{2k}}\mathbf{w}_{\tau_z+s}^T\mathbf{w}_{\tau_z+s}\lambda_{\max}((\mathbf{A}^{k-1-s})^T\mathbf{A}^{k-1-s})\right)^{\frac{2+\epsilon}{2}} \\
&\leq \sum_{s=0}^{k-1}(n^2k^{2n}\mathbf{w}_{\tau_z+s}^T\mathbf{w}_{\tau_z+s})^{\frac{2+\epsilon}{2}} \\
&= n^{2+\epsilon}k^{n(2+\epsilon)}\sum_{s=0}^{k-1}\left(\sum_{i=1}^{n}(w_{\tau_z+s}^i)^2\right)^{\frac{2+\epsilon}{2}} \\
&\leq n^{2+\epsilon}k^{n(2+\epsilon)}\sum_{s=0}^{k-1}n^{\frac{\epsilon}{2}}\sum_{i=1}^{n}|w_{\tau_z+s}^i|^{2+\epsilon} \qquad (3.9) \\
&= n^{2+\frac{3}{2}\epsilon}k^{n(2+\epsilon)}\sum_{s=0}^{k-1}\sum_{i=1}^{n}|w_{\tau_z+s}^i|^{2+\epsilon}.
\end{aligned}$$

In (3.9) we have again used Jensen's inequality and this gives the desired bound.

Now, combining (3.4) with (3.8) gives the bound

$$\bar{P}_z(k) \leq C\frac{k^{(n+1)(\epsilon+2)}}{((\rho')^{2+\epsilon})^{k-1}}. \tag{3.10}$$

Note that to obtain Theorems 2.4.6 and 2.4.7, we require for some H that

$$Q(k)\bar{P}_z(k)\rho^{2k}|\lambda|^{2k} \leq d^k, \tag{3.11}$$

for all $k > H$ where $Q(k)$ is any finite polynomial in k and $d < 1$. We also see from the proof of Theorem 2.4.5 that for any $1 < \rho' < \rho$ there exists an H such that (3.10) holds for all $k > H$. Let us write $\rho' = \zeta\rho$ where $0 < \zeta < 1$ and note that ζ can be made arbitrarily close to 1. It then follows that

$$Q(k)\bar{P}_z(k)\rho^{2k}|\lambda|^{2k} \leq Q(k)C\frac{k^{(n+1)(\epsilon+2)}}{((\rho')^{2+\epsilon})^{k-1}}\rho^{2k}|\lambda|^{2k} = C'Q'(k)\Big(\frac{\rho^2|\lambda|^2}{\zeta^{2+\epsilon}\rho^2\rho^\epsilon}\Big)^k$$
$$= C'Q'(k)\Big(\frac{|\lambda|^2}{\zeta^{2+\epsilon}\rho^\epsilon}\Big)^k,$$

where we have defined $C' = C(\rho')^{2+\epsilon}$ and $Q'(k) = Q(k)k^{(n+1)(\epsilon+2)}$. Thus, given an $\epsilon > 0$, we can choose ρ sufficiently large so that $\rho^\epsilon > |\lambda|^2$ and ζ close enough to 1 so that

$$d' = \frac{|\lambda|^2}{\zeta^{2+\epsilon}\rho^\epsilon} < 1.$$

Now, to obtain (3.11), we need only show that there exists an H such that

$$C'Q'(k)\Big(\frac{d'}{d}\Big)^k < 1$$

holds for all $k > H$ and for some $d < 1$. Pick any $d' < d < 1$. Then, as in the proof of Theorem 2.4.5, L'Hôpital's rule shows that the left term above goes to zero as $k \to \infty$. Thus, (3.11) holds for some $d < 1$. All supporting results from Chapter 2 now follow the proofs in the Gaussian case identically and we obtain the result. In particular, the fact that the noise distributions have finite second moments implies Theorem 2.4.7. The assumption that $\mathcal{Z}$ admits a density which is positive on every open set gives irreducibility and allows the application of our drift criteria. □

Chapter 4

Multi-Sensor Systems

In this chapter, we consider multi-sensor, single-controller systems.

4.1 Problem Statement

Consider the class of multi-sensor LTI discrete-time systems with both plant and observation noise. The system equations are given by

$$\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{B}\mathbf{u}_t + \mathbf{w}_t, \quad \mathbf{y}_t^j = \mathbf{C}^j\mathbf{x}_t + \mathbf{v}_t^j, \quad 1 \le j \le M, \tag{4.1}$$

where $\mathbf{x}_t \in \mathbb{R}^n$ and $\mathbf{u}_t \in \mathbb{R}^m$ are the state and control action variables at time $t \in \mathbb{N}$ respectively. The observation made by sensor j at time t is denoted by $\mathbf{y}_t^j \in \mathbb{R}^{p_j}$. The matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}^j$ and random vectors $\mathbf{w}_t, \mathbf{v}_t^j$ are of compatible size.

We require that $\{\mathbf{w}_t\}$ and each $\{\mathbf{v}_t^j\}$ be sequences of i.i.d. random vectors drawn from a distribution $\mathcal{Z}$, with finite $2+\epsilon$ moments in each component for some $\epsilon > 0$, which admits a probability density that is positive on every open set. At time t, $\mathbf{w}_t$ and each $\mathbf{v}_t^j$ are independent of each other and the state $\mathbf{x}_t$. The initial state, $\mathbf{x}_0$, is drawn from the $\mathcal{Z}$ distribution.

Assumption 4.1.1. *We require controllability and joint observability. That is, the pair* $(\mathbf{A}, \mathbf{B})$ *is controllable and the pair* $([(\mathbf{C}^1)^T \quad \cdots \quad (\mathbf{C}^M)^T]^T, \mathbf{A})$ *is observable but the individual pairs* $(\mathbf{C}^j, \mathbf{A})$ *may not be observable.*

The setup is depicted in Figure 4.1. The observations are made by a set of M sensors and each sensor sends information to the controller through a finite capacity channel. At each time stage t, we allow sensor $j \in \{1, \ldots, M\}$ to send an encoded value $q_t^j \in \{1, 2, \ldots, N_t^j\}$ for some $N_t^j \in \mathbb{N}$. In addition, the controller can send a feedback value $b_t \in \{0, 1\}$ at times $t = Ts$, where T is the period of our coding policy and $s \in \mathbb{N}$. The value b_t is seen by all sensors at time t. We define the rate at time t as $R_t = \sum_{j=1}^{M} \log_2(N_t^j)$. The coding scheme is applied periodically with period T and so the rate for all time stages is specified by $\{N_0^j, \ldots, N_{T-1}^j : 1 \leq j \leq M\}$. The average rate is

$$R_{\text{avg}} = \frac{1}{T}\left(M + \sum_{t=0}^{T-1} R_t\right), \tag{4.2}$$

accounting for the encoded and feedback values.

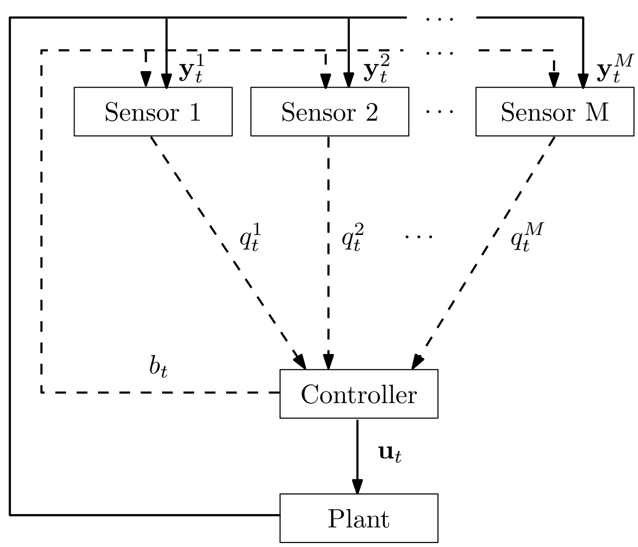

Figure 4.1: A multi-sensor system with finite-rate communication channels.

The controllability matrix of the controller is $\mathcal{C}_{(\mathbf{A},\mathbf{B})} = \begin{bmatrix}\mathbf{B} & \mathbf{AB} & \cdots & \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix}$. We denote the controllable subspace of the controller by K, which is the range space of $\mathcal{C}_{(\mathbf{A},\mathbf{B})}$. The observability matrix of sensor j is

$$\mathcal{O}_{(\mathbf{C}^j,\mathbf{A})} = \begin{bmatrix}(\mathbf{C}^j)^T & (\mathbf{C}^j\mathbf{A})^T & \cdots & (\mathbf{C}^j\mathbf{A}^{n-1})^T\end{bmatrix}^T,$$

the null space is $N^j = \mathrm{Ker}(\mathcal{O}_{(\mathbf{C}^j,\mathbf{A})})$ and the observable subspace is defined to be $O^j = (N^j)^\perp$ for $1 \le j \le M$.

Information structure. For a process $\{\mathbf{x}_t\}$ we define $\mathbf{x}_{[a,b]} = \{\mathbf{x}_a, \mathbf{x}_{a+1}, \dots, \mathbf{x}_b\}$. At time t, each sensor j maps its information $I_t^{s_j} := \{\mathbf{y}^j_{[0,t]}, b_{[0,t-1]}\} \to q_t^j \in \{1, \dots, N_t^j\}$. The controller maps its information $I_t^c := \{q^1_{[0,t]}, \dots, q^M_{[0,t]}\} \to (\mathbf{u}_t, b_t) \in \mathbb{R}^m \times \{0,1\}$.

4.2 Main Result

Let $V_1, \dots, V_\ell$ denote the generalized eigenspaces of $\mathbf{A}$. We state the following assumption. In Section 4.3 we will remove this assumption and consider the general case.

Assumption 4.2.1. *Each eigenspace is observed by some sensor. That is, for each* $1 \le i \le \ell$ *there exists a* $1 \le j \le M$ *such that* $V_i \subseteq O^j$.

We label the eigenvalues of $\mathbf{A}$ as $\lambda_1, \dots, \lambda_n$. Without loss of generality, we assume that $|\lambda_i| > 1$ for all $1 \le i \le n$. Our main result for multi-sensor systems is the following:

Theorem 4.2.2. *Under Assumption 4.2.1, there exists a coding and control policy with average rate* $R_{avg} \le 1/(T2n)(M + \sum_{i=1}^n \log_2(\lceil |\lambda_i|^{T2n} + \epsilon \rceil + 1))$ *for some* $\epsilon > 0$ *which gives:*

(a) the existence of a unique invariant distribution for $\{\mathbf{x}_{2nt}\}$;

(b) $\lim_{t\to\infty} E[\|\mathbf{x}_{2nt}\|_2^2] < \infty$.

Theorem 4.2.3. *The average rate in Theorem 4.2.2 achieves the minimum rate* (1.9) *asymptotically for large sampling periods. That is,* $\lim_{T\to\infty} R_{avg} = R_{\min}$.

Proof of Theorem 4.2.3: Follows from the proof of Theorem 2.2.2. □

The proof of Theorem 4.2.2 is basically an application of the Jordan normal form together with Assumption 4.2.1.

Proof of Theorem 4.2.2: Under Assumption 4.2.1, we can assign each eigenspace $V_i \subseteq O^j$ to some sensor j. Let $V_{j,1}, \dots, V_{j,m_j}$ denote the eigenspaces assigned to sensor j and let us write $V_{j,i} = \mathrm{span}\{\mathbf{v}_{j,i,1}, \dots, \mathbf{v}_{j,i,d_{j,i}}\}$ where each $\mathbf{v}_{j,i,h} \in \mathbb{R}^{1\times n}$ is a generalized eigenvector. We put

$$\mathbf{Q}_{j,i} = \begin{bmatrix} (\mathbf{v}_{j,i,1})^T & \cdots & (\mathbf{v}_{j,i,d_{j,i}})^T \end{bmatrix}^T,$$
$$\mathbf{Q}_j = \begin{bmatrix} \mathbf{Q}_{j,1}^T & \cdots & \mathbf{Q}_{j,m_j}^T \end{bmatrix}^T,$$
$$\mathbf{Q} = \begin{bmatrix} (\mathbf{Q}_M)^T & \cdots & (\mathbf{Q}_1)^T \end{bmatrix}^T.$$

We apply the similarity transform $\bar{\mathbf{x}}_t = \mathbf{Q}\mathbf{x}_t$ to (4.1) and define $\bar{\mathbf{A}} = \mathbf{Q}\mathbf{A}\mathbf{Q}^{-1}$, $\bar{\mathbf{B}} = \mathbf{Q}\mathbf{B}$ and $\bar{\mathbf{w}}_t = \mathbf{Q}\mathbf{w}_t$ to get the system

$$\bar{\mathbf{x}}_{t+1} = \bar{\mathbf{A}}\bar{\mathbf{x}}_t + \bar{\mathbf{B}}\bar{\mathbf{u}}_t + \bar{\mathbf{w}}_t. \tag{4.3}$$

We now look at the estimation of the state by the sensors. For convenience, let us write

$$\bar{\mathbf{x}}_t = \begin{bmatrix} (\bar{\mathbf{x}}_t^M)^T & \cdots & (\bar{\mathbf{x}}_t^1)^T \end{bmatrix}^T,$$
$$\bar{\mathbf{x}}_t^j = \begin{bmatrix} (\bar{\mathbf{x}}_t^{j,1})^T & \cdots & (\bar{\mathbf{x}}_t^{j,m_j})^T \end{bmatrix}^T,$$
$$\bar{\mathbf{x}}_t^{j,i} = \begin{bmatrix} \bar{x}_t^{j,i,1} & \cdots & \bar{x}_t^{j,i,d_{j,i}} \end{bmatrix}^T,$$

with $\bar{x}_t^{j,i,h} \in \mathbb{R}$. Let us write

$$\mathcal{O}_{(\mathbf{C}^j,\mathbf{A})} = \begin{bmatrix} (\mathbf{o}_{j,1})^T & \cdots & (\mathbf{o}_{j,np_j})^T \end{bmatrix}^T$$

where each $\mathbf{o}_{j,i} \in \mathbb{R}^{1\times n}$.

With our construction above, under Assumption 4.2.1, we have, for each j, i, h, that $\mathbf{v}_{j,i,h} = \sum_{\ell=1}^{np_j} k_\ell^{j,i,h}\mathbf{o}_{j,\ell}$ for some real coefficients $\{k_\ell^{j,i,h}\}$. Consider the first n time stages. By putting $\mathbf{k}^{j,i,h} = \begin{bmatrix} k_1^{j,i,h} & \cdots & k_{np_j}^{j,i,h} \end{bmatrix}$, it follows that

$$\mathbf{k}^{j,i,h} \begin{bmatrix} (\mathbf{y}_0^j)^T & \cdots & (\mathbf{y}_{n-1}^j)^T \end{bmatrix}^T = \mathbf{k}^{j,i,h}\mathcal{O}_{(\mathbf{C}^j,\mathbf{A})}\mathbf{x}_0 + \bar{v}_0^{j,i,h}$$
$$= \sum_{\ell=1}^{np_j} k_\ell^{j,i,h}\mathbf{o}_{j,\ell}\mathbf{x}_0 + \bar{v}_0^{j,i,h} = \mathbf{v}_{j,i,h}\mathbf{x}_0 + \bar{v}_0^{j,i,h} = \bar{x}_0^{j,i,h} + \bar{v}_0^{j,i,h},$$

where $\bar{v}_0^{j,i,h}$ is some noise term. We will use the same notation for $\bar{\mathbf{v}}_0^j$ that we use for $\bar{\mathbf{x}}_t^j$.

As in Lemma 2.4.1 for the single-station case, we can use the next n times stages to apply a control action. We then apply the above scheme repeatedly and sample every $2n$ time stages.

Furthermore, since $\mathbf{Q}$ is the Jordan normal transformation matrix, it follows that $\bar{\mathbf{A}}^{2n} = \text{diag}(\bar{\mathbf{J}}_1^{2n}, \dots, \bar{\mathbf{J}}_\ell^{2n})$ where each $\bar{\mathbf{J}}_i \in \mathbb{R}^{d_i \times d_i}$ is a Jordan block. Since we can apply another Jordan transformation to this sampled system, we can assume without loss of generality that $\bar{\mathbf{A}}^{2n}$ is actually in Jordan form and each $\bar{\mathbf{J}}_i^{2n}$ is actually a Jordan block.

To simplify notation, we write $\mathbf{A} := \bar{\mathbf{A}}^{2n} = \text{diag}(\mathbf{J}_1, \dots, \mathbf{J}_\ell)$ where each $\mathbf{J}_i \in \mathbb{R}^{d_i \times d_i}$ is a Jordan block, $\mathbf{x}_s = \begin{bmatrix} (\mathbf{x}_s^1)^T & \cdots & (\mathbf{x}_s^\ell)^T \end{bmatrix}^T$ where $\mathbf{x}_s^i \in \mathbb{R}^{d_i}$ and similarly for $\mathbf{u}_s, \mathbf{w}_s, \mathbf{y}_s$ and $\mathbf{v}_s$. From the above, we can also see that for each i, there exists a j such that $\mathbf{x}_s^i + \mathbf{v}_s^i$ is known by sensor j at time s.

Thus our system is equivalent to the following subsystems:

$$\mathbf{x}_{s+1}^i = \mathbf{J}_i \mathbf{x}_s^i + \mathbf{u}_s^i + \mathbf{w}_s^i, \qquad \mathbf{y}_s^i = \mathbf{x}_s^i + \mathbf{v}_s^i, \qquad 1 \le i \le \ell,$$

where, for each $1 \le i \le \ell$, there exists a sensor j which knows $\mathbf{y}_s^i$ at time s and $\mathbf{u}_s^i$ is chosen arbitrarily by the controller.

As in Section 2.3, we let Δ_s be the vector of bin sizes at time s and define the sequence of stopping times

$$\tau_0 = 0, \tau_{z+1} = \min\{s > \tau_z : |\mathbf{y}_s| = |\mathbf{x}_s + \mathbf{v}_s| \le \Delta_s\}.$$

The feedback value b_{2ns} is chosen as

$$b_{2ns} = \begin{cases} 1, & \text{if } s = \tau_z \text{ for some } z \in \mathbb{N}, \\ 0, & \text{otherwise,} \end{cases}$$

so that the coding and control policy as in Section 2.3 is implementable at the sensors and at the controller. This reduces the problem to the single-station case and we obtain the result. □

4.3 Sufficient Conditions for the General Multi-Sensor Case

The following theorem extends the classical observability canonical decomposition to the decentralized case. For a detailed proof in the centralized case, see [24]. The more general multi-agent setup, where each agent makes observations and applies a control action, can be found in [15]. We are not aware of an explicit proof for our case and we give a proof of Theorem 4.3.1 in this section for the convenience of the reader.

Theorem 4.3.1. *Under Assumption 4.1.1, there exists a matrix* $\mathbf{Q}$ *such that if we define* $\bar{\mathbf{A}} = \mathbf{Q}\mathbf{A}\mathbf{Q}^{-1}$ *and* $\bar{\mathbf{C}}^j = \mathbf{C}^j\mathbf{Q}^{-1}$*, then*

$$\bar{\mathbf{A}} = \begin{bmatrix} \bar{\mathbf{A}}_M & * & \cdots & * \\ & \bar{\mathbf{A}}_{M-1} & \cdots & * \\ & & \ddots & \\ & 0 & & \bar{\mathbf{A}}_1 \end{bmatrix}, \tag{4.4a}$$

$$\begin{bmatrix} \bar{\mathbf{C}}^M \\ \bar{\mathbf{C}}^{M-1} \\ \vdots \\ \bar{\mathbf{C}}^1 \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{C}}_O^M & * & \cdots & * \\ & \bar{\mathbf{C}}_O^{M-1} & \cdots & * \\ & & \ddots & \\ & 0 & & \bar{\mathbf{C}}_O^1 \end{bmatrix}, \tag{4.4b}$$

where the $*$*'s denote irrelevant submatrices, each* $\bar{\mathbf{A}}_j \in \mathbb{R}^{n_j \times n_j}$ *and each* $\bar{\mathbf{C}}_O^j \in \mathbb{R}^{p_j \times n_j}$*.*

Proof of Theorem 4.3.1: We define $n_1 = \dim(O^1)$, and $n_j = \dim(O^j) - \dim(O^j \cap (\cup_{i=1}^{j-1} O^i))$, for $2 \leq j \leq M$. We choose n_1 linearly independent row vectors from $\mathcal{O}_{(\mathbf{C}^1,\mathbf{A})}$ and label them $\mathbf{q}_1^1, \ldots, \mathbf{q}_{n_1}^1$. Proceeding by induction, we choose $\{\mathbf{q}_1^j, \ldots, \mathbf{q}_{n_j}^j\}$ from $\mathcal{O}_{(\mathbf{C}^j,\mathbf{A})}$ such that

$$\{\mathbf{q}_1^1, \ldots, \mathbf{q}_{n_1}^1, \mathbf{q}_1^2, \ldots, \mathbf{q}_{n_2}^2, \ldots, \mathbf{q}_1^j, \ldots, \mathbf{q}_{n_j}^j\}$$

is a set of linearly independent vectors.

We define

$$\mathbf{Q}^j = \begin{bmatrix} (\mathbf{q}_1^j)^T & \cdots & (\mathbf{q}_{n_j}^j)^T \end{bmatrix}^T$$

for all $1 \leq j \leq M$ and concatenate these matrices to choose our transformation matrix

$$\mathbf{Q} = \begin{bmatrix} (\mathbf{Q}^M)^T & \cdots & (\mathbf{Q}^1)^T \end{bmatrix}^T.$$

It will also be convenient to denote the rows of $\mathbf{Q}$ by $\mathbf{q}_1, \ldots, \mathbf{q}_n$ so that $\mathbf{Q} = \begin{bmatrix} (\mathbf{q}_1)^T & \cdots & (\mathbf{q}_n)^T \end{bmatrix}^T$.

From the Cayley–Hamilton Theorem, we know that for all $m \geq n$ there exist $\alpha_0, \ldots, \alpha_{n-1}$ such that $\mathbf{A}^m = \sum_{i=0}^{n-1} \alpha_i \mathbf{A}^i$. Since $\{\mathbf{q}_1^j, \ldots, \mathbf{q}_{n_j}^j\}$ are rows of $\mathcal{O}_{(\mathbf{C}^j, \mathbf{A})}$, this implies that $\mathbf{q}_i^j \mathbf{A}$ is in the row space of $\mathcal{O}_{(\mathbf{C}^j, \mathbf{A})}$ for all $1 \leq i \leq n_j$. Let us define the sets

$$S_j := \{\mathbf{q}_1^1, \ldots, \mathbf{q}_{n_1}^1, \mathbf{q}_1^2, \ldots, \mathbf{q}_{n_2}^2, \ldots, \mathbf{q}_1^j, \ldots, \mathbf{q}_{n_j}^j\}, \qquad 1 \leq j \leq M.$$

From our construction, it is then clear that

$$\mathbf{q}_1^j \mathbf{A}, \ldots, \mathbf{q}_{n_j}^j \mathbf{A} \in \operatorname{span}(S_j). \tag{4.5}$$

We write $\bar{\mathbf{A}}$ in terms of its column vectors as $\bar{\mathbf{A}} = \begin{bmatrix} \bar{\mathbf{a}}_1 & \cdots & \bar{\mathbf{a}}_n \end{bmatrix}$ where $\bar{\mathbf{a}}_i \in \mathbb{R}^{n \times 1}$ for each $1 \leq i \leq n$. Our similarity transform gives

$$\bar{\mathbf{A}}\mathbf{Q} = \mathbf{Q}\mathbf{A}. \tag{4.6}$$

Recall from linear algebra that we can write the left side of (4.6) as $\sum_{i=1}^n \bar{\mathbf{a}}_i \mathbf{q}_i$ where $\bar{\mathbf{a}}_i \mathbf{q}_i \in \mathbb{R}^{n \times n}$ for each $1 \leq i \leq n$. Now, to return to our earlier notation, each vector $\mathbf{q}_i^j \mathbf{A}$ is a linear combination of $\{\mathbf{q}_\ell^k : 1 \leq k \leq j, 1 \leq \ell \leq n_k\}$ and is linearly independent of the remaining rows of $\mathbf{Q}$. Since $\mathbf{QA} = \begin{bmatrix} (\mathbf{q}_1\mathbf{A})^T & \cdots & (\mathbf{q}_n\mathbf{A})^T \end{bmatrix}^T$, we see from (4.6) that the i^{th} row of $\bar{\mathbf{A}}$ is the representation of $\mathbf{q}_i \mathbf{A}$ with respect to $\mathbf{q}_1, \ldots, \mathbf{q}_n$. More precisely, we write $\bar{\mathbf{a}}_i = \begin{bmatrix} \bar{a}_{i,1} & \cdots & \bar{a}_{i,n} \end{bmatrix}^T$ with each $\bar{a}_{i,h} \in \mathbb{R}$ so that (4.6) gives the system of equations

$$\sum_{i=1}^n \bar{a}_{i,h} \mathbf{q}_i = \mathbf{q}_h \mathbf{A}, \quad 1 \leq h \leq n. \tag{4.7}$$

Combining (4.5) and (4.7) gives the desired form.

We next turn our attention to the form of $\bar{\mathbf{C}}^j$. Since each $\mathbf{C}^j$ is a submatrix of $\mathcal{O}_{(\mathbf{C}^j,\mathbf{A})}$, it is clear that the rows of $\mathbf{C}^j$ are in the span of S_j. Since $\bar{\mathbf{C}}^j\mathbf{Q} = \mathbf{C}^j$, by writing $\bar{\mathbf{C}}^j$ in terms of its column vectors we obtain the desired form. □

Remark 4.3.2. *In the proof of Theorem 4.3.1, we give one construction for the triangular decomposition in* (4.4)*. This transformation is not unique. There may be many ways to achieve a block upper triangular form and it is not necessary to place the sensors in order $M, \ldots, 1$.*

In Section 4.2, Assumption 4.2.1 allowed us to reduce the system to a set of subsystems. Without this assumption, the lower components of the state act as noise for the upper components. In particular, we need to bound these lower modes when all quantizers are perfectly-zoomed to achieve Theorem 2.4.5. To do this, we must have that the bin sizes of the lower modes are small compared with the upper ones as is needed in equation (4.11) for example. With many different eigenvalues, we cannot guarantee this in the general case. Below, we give a sufficient rate and an alternative assumption for stability.

For Theorem 4.3.3 below, let us write $\Lambda(\bar{\mathbf{A}}_j) = \{\lambda_{j,1}, \ldots, \lambda_{j,n_j}\}$ where $\bar{\mathbf{A}}_j$ is given in (4.4).

Theorem 4.3.3. *There exists a coding and control policy which gives:*

(a) the existence of a unique invariant distribution for $\{\mathbf{x}_{2nt}\}$;

(b) $\lim_{t\to\infty} E[\|\mathbf{x}_{2nt}\|_2^2] < \infty$,

and with average rate in the limit of large sampling periods

$$\lim_{T\to\infty} R_{avg} = \sum_{j=1}^{M}\sum_{i=1}^{n_j} \log_2(\max\{|\lambda_{j,i}|, |\lambda_{h,\ell}| : h < j, 1 \le \ell \le n_h\}).$$

Clearly, we could also achieve (a) and (b) in Theorem 4.3.3 with $\lim_{T\to\infty} R_{\text{avg}} = n\log_2(\lambda_{\text{absmax}})$ where $\lambda_{\text{absmax}} = \max_{j,i}\{|\lambda_{j,i}|\}$.

Informally, Theorem 4.3.3 tells us that in order to stabilize a component, we must apply a rate capable of stabilizing any mode that is driving that component.

For Theorem 4.3.4 below, recall that we have some flexibility in the decomposition given by Theorem 4.3.1. See the proof of Theorem 4.3.1 and Remark 4.3.2.

Theorem 4.3.4. *If the eigenvalues of* $\bar{\mathbf{A}}_M, \ldots, \bar{\mathbf{A}}_1$ *in* (4.4) *are ordered in decreasing magnitude then Theorem 4.2.2 holds without Assumption 4.2.1. That is, the theorem holds if for* $\lambda_i \in \Lambda(\bar{\mathbf{A}}_i)$ *and* $\lambda_j \in \Lambda(\bar{\mathbf{A}}_j)$ *we have that* $|\lambda_i| \leq |\lambda_j|$ *when* $i < j$.

4.4 Coding and Control Policy for the General Multi-Sensor Case

Consider the system (4.1). Sampling, observing and controlling as in the proof of Theorem 2.2.1 and applying the transformation $\bar{\mathbf{x}}_t = \mathbf{Q}\mathbf{x}_t$ where $\mathbf{Q}$ is given in Theorem 4.3.1, we obtain the system

$$\mathbf{x}_{s+1} = \mathbf{A}\mathbf{x}_s + \mathbf{u}_s + \mathbf{w}_s, \qquad \mathbf{y}_s = \mathbf{x}_s + \mathbf{v}_s.$$

We do not relabel the variables (for example $\bar{\mathbf{x}}_s$) by a slight abuse of notation and for the sake of reasonable presentation.

In the above, $\mathbf{A}$ is block upper triangular with the blocks $\mathbf{A}_M, \ldots, \mathbf{A}_1$ descending along the diagonal and each $\mathbf{A}_j \in \mathbb{R}^{n_j \times n_j}$ as in (4.4a) of Theorem 4.3.1. Since we can always apply a block transformation to $\mathbf{A}$ in which each of the blocks is the Jordan tranformation of $\mathbf{A}_j$, we can assume without loss of generality that each $\mathbf{A}_j$ is in real Jordan normal form and we write $\mathbf{A}_j = \mathrm{diag}(\mathbf{A}_{j,1}, \ldots, \mathbf{A}_{j,m_j})$ where each $\mathbf{A}_{j,i} \in \mathbb{R}^{d_{j,i} \times d_{j,i}}$ is a Jordan block.

Let us write $\mathbf{x}_s = \begin{bmatrix} (\mathbf{x}_s^M)^T & \cdots & (\mathbf{x}_s^1)^T \end{bmatrix}^T$ where $\mathbf{x}_s^j = \begin{bmatrix} (\mathbf{x}_s^{j,1})^T & \cdots & (\mathbf{x}_s^{j,m_j})^T \end{bmatrix}^T$ and $\mathbf{x}_s^j \in \mathbb{R}^{n_j}$ with $\mathbf{x}_s^{j,i} = \begin{bmatrix} x_s^{j,i,1} & \cdots & x_s^{j,i,d_{j,i}} \end{bmatrix}^T$ and each $x_s^{j,i,h} \in \mathbb{R}$. We will use the same notational convention for all relevant vectors in this section. Namely, we will follow this convention for $\mathbf{u}_s$, $\mathbf{w}_s$, $\mathbf{y}_s$, $\mathbf{v}_s$ and for Δ_s, $\hat{\mathbf{x}}_s$, $\mathbf{L}$ which will be specified.

From the proof of Theorem 4.3.1, we know that the rows of $\mathbf{Q}$ are taken from the row spaces of $\{\mathcal{O}_{(\mathbf{C}^j, \mathbf{A})}\}$ and we can see how $\mathbf{y}_s^j$ is known by sensor j at time s.

Let us denote the eigenvalue of $\mathbf{A}_{j,i}$ by $\lambda_{j,i}$. We define

$$|\lambda'_{j,i}| = \max\{|\lambda_{j,i}|, |\lambda_{h,\ell}| : 1 \leq h < j, 1 \leq \ell \leq m_h\}.$$

Let $K_{j,i} = \lceil |\lambda'_{j,i}| + \delta + \epsilon \rceil$ for some $\delta, \epsilon > 0$. Let $\mathbf{K}_j = \mathrm{diag}(K_{j,1}\mathbf{I}, \ldots, K_{j,m_j}\mathbf{I})$ where each $\mathbf{I}$ is the identity matrix of appropriate size so that $K_{j,i}\mathbf{I} \in \mathbb{R}^{d_{j,i}\times d_{j,i}}$. Let $\mathbf{K} = \operatorname{diag}(\mathbf{K}_M, \ldots, \mathbf{K}_1)$.

Let $\Delta_s^{j,i,h}$ be the bin size corresponding to the component $x_s^{j,i,h}$ at time s. We let $q_s^{j,i,h} = \mathcal{E}_{K_{j,i}}^{\Delta_s^{j,i,h}}(y_s^{j,i,h})$. Let our fixed rate for sensor j be $N^j = (\prod_{i=1}^{m_j} K_{j,i}^{d_{j,i}}) + 1$ for all $s \in \mathbb{N}$. Choose any invertible function $f_j : \prod_{i=1}^{m_j}\{1, \ldots, K_{j,i}\}^{d_{j,i}} \to \{1, \ldots, \prod_{i=1}^{m_j} K_{j,i}^{d_{j,i}}\}$. We then choose the encoded value

$$q_s^j = \begin{cases} f_j(q_s^{j,1,1}, \ldots, q_s^{j,m_j,d_{j,m_j}}), & \text{if } q_s^{j,i,h} \neq 0 \text{ for all } 1 \leq i \leq m_j,\ 1 \leq h \leq d_{j,i}, \\ 0, & \text{otherwise.} \end{cases}$$

Upon receiving $q_s^j \neq 0$, the controller knows

$$\{q_s^{j,i,h} : 1 \leq i \leq m_j, 1 \leq h \leq d_{j,i}\}.$$

The controller forms the estimate $\hat{\mathbf{x}}_s^j$ as

$$\hat{\mathbf{x}}_s^j = \begin{bmatrix} (\hat{\mathbf{x}}_s^{j,1})^T & \cdots & (\hat{\mathbf{x}}_s^{j,m_j})^T \end{bmatrix}^T,$$

where

$$\hat{\mathbf{x}}_s^{j,i} = \begin{bmatrix} \hat{x}_s^{j,i,1} & \cdots & \hat{x}_s^{j,i,d_{j,i}} \end{bmatrix}^T$$

and

$$\hat{x}_s^{j,i,h} = \begin{cases} \mathcal{D}_{K_{j,i}}^{\Delta_s^{j,i,h}}(q_s^{j,i,h}), & \text{if } q_s^1, \ldots, q_s^M \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Recall that the feedback bit b_s is used to coordinate the sensors, making our control policy implementable. It is chosen as

$$b_s = \begin{cases} 1, & \text{if } q_s^1, \ldots, q_s^M \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

The update equations are

$$\Delta_{s+1}^{j,i} = \bar{Q}^{j,i}\left(b_s, \Delta_s^{j,i}\right)\Delta_s^{j,i}, \quad \bar{Q}^{j,i}\left(b_s, \Delta_s^{j,i}\right) = \begin{cases} \rho(|\lambda'_{j,i}| + \delta), & \text{if } b_s = 0, \\ \beta_{j,i}(\Delta_s^{j,i}), & \text{otherwise,} \end{cases} \tag{4.8}$$

for some $\rho > 1$ and with

$$\beta_{j,i}(\Delta_s^{j,i}) = \text{diag}(\beta_{j,i,1}(\Delta_s^{j,i,1}), \ldots, \beta_{j,i,d_{j,i}}(\Delta_s^{j,i,d_{j,i}})), \tag{4.9a}$$

$$\beta_{j,i,h}(\Delta_s^{j,i,h}) = \begin{cases} 1, & \text{if } \Delta_s^{j,i,h} \leq L^{j,i,h} , \\ \frac{|\lambda'_{j,i}|+\delta}{|\lambda'_{j,i}|+\delta+\epsilon-\eta}, & \text{otherwise,} \end{cases} \tag{4.9b}$$

for some $0 < \eta < \epsilon$ and $\mathbf{L}^{j,i} \in \mathbb{R}_+^{d_{j,i}}$. Note that if we define $\bar{\mathbf{L}}^{j,i} = \mathbf{L}^{j,i}(|\lambda'_{j,i}|+\delta)/(|\lambda'_{j,i}|+\delta+\epsilon-\eta)$ then $\Delta_s > \bar{\mathbf{L}}$ for all $s \in \mathbb{N}$.

Bin ordering. From (4.4a) of Theorem 4.3.1, we can write

$$\mathbf{A} = \begin{bmatrix}(\mathbf{A}_M^R)^T & \cdots & (\mathbf{A}_1^R)^T\end{bmatrix}^T$$

where each $\mathbf{A}_j^R \in \mathbb{R}^{n_j \times n}$. We can further write $\mathbf{A}_j^R = \begin{bmatrix}0 & \mathbf{A}_j & \mathbf{M}_j\end{bmatrix}$ where

$$\mathbf{A}_j = \text{diag}(\mathbf{A}_{j,1}, \ldots, \mathbf{A}_{j,m_j})$$

and

$$\mathbf{M}_j = \begin{bmatrix}(\mathbf{M}_{j,1})^T & \cdots & (\mathbf{M}_{j,m_j})^T\end{bmatrix}^T$$

with

$$\mathbf{M}_{j,i} = \begin{bmatrix}\mathbf{M}_{j,i,j-1} & \cdots & \mathbf{M}_{j,i,1}\end{bmatrix}$$

and each $\mathbf{M}_{j,i,h} \in \mathbb{R}^{d_{j,i}, n_h}$. Recall from the proof of Theorem 4.3.1 that $n_h = \dim(O^h) - \dim(O^h \cap (\cup_{i=1}^{h-1} O^i))$. While the notation is complicated, the decomposition is simple and we illustrate the above in Figure 4.2.

Let us denote the entries of $\mathbf{M}_{j,i,h}$ by $\{m_{j,i,h}^{k,\ell}\}$. We define the entry of maximum absolute value as

$$\kappa_{j,i,h} = \max_{k,\ell}\{|m_{j,i,h}^{k,\ell}|\}.$$

We set $\mathbf{L} = c\Delta_0$, for some $0 < c \leq 1$. For any $\delta > 0$, by our coding and control policy (and in particular the choice of $\{|\lambda'_{j,i}|\}$) given above, we can choose Δ_0 such that the following ordering is maintained for all $s \in \mathbb{N}$:

$$\Delta_s^{j,i,h+1} \leq \frac{\delta}{j}\Delta_s^{j,i,h}, \tag{4.10a}$$

$$
\begin{aligned}
&\qquad 1 \le j \le M,\ 1 \le i \le m_j,\ 1 \le h \le d_{j,i}-1, \\
&\Delta_s^{j,k,1} \le \Delta_s^{j,i,1}, && (4.10b)\\
&\qquad 1 \le j \le M,\ 1 \le i \le m_j - 1,\ i < k \le m_j, \\
&\Delta_s^{k,\ell,h} \le \frac{\delta}{j n_k \kappa_{j,i,k}} \Delta_s^{j,i,d_{j,i}}, && (4.10c)\\
&\qquad 2 \le j \le M,\ 1 \le i \le m_j,\ 1 \le k \le j-1,\ 1 \le \ell \le m_k,\ 1 \le h \le d_{k,\ell}.
\end{aligned}
$$

Informally, we order the bins within Jordan blocks $\mathbf{A}_{j,i}$, within sensor blocks $\mathbf{A}_j$ and between sensor blocks $\mathbf{A}_j$.

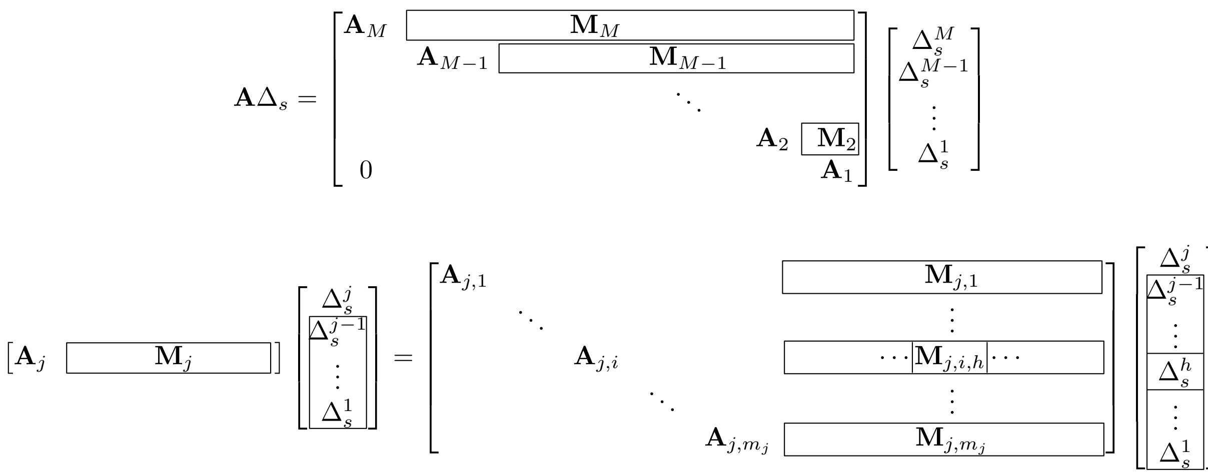

Figure 4.2: Illustration of the notation used to describe $\mathbf{A}$.

Control action. We choose the control action

$$\mathbf{u}_s = -\mathbf{A}\hat{\mathbf{x}}_s.$$

We define the sequence of stopping times

$$\tau_0 = 0, \quad \tau_{z+1} = \min\left\{ s > \tau_z : |\mathbf{y}_s| = |\mathbf{x}_s + \mathbf{v}_s| \le \frac{1}{2}\mathbf{K}\Delta_s \right\}.$$

Proof of Theorem 4.3.3: Let $\lambda \in \mathbb{R}$. The proof is similar for $\lambda \in \mathbb{C}$. Let us define

$$\tilde{\mathbf{A}}_{j,i} = \left[(\tilde{\mathbf{A}}_{j,i}^R)^T \quad (\tilde{\mathbf{A}}_{j-1}^R)^T \quad \cdots \quad (\tilde{\mathbf{A}}_1^R)^T \right]^T$$

where

$$\tilde{\mathbf{A}}^R_{j,i} = \begin{bmatrix} \mathbf{A}_{j,i} & \mathbf{M}_{j,i,j-1} & \cdots & \mathbf{M}_{j,i,1} \end{bmatrix}$$

and each $\tilde{\mathbf{A}}^R_j$ is just $\mathbf{A}^R_j$ with a sufficient number of leading zero columns removed to make $\tilde{\mathbf{A}}_{j,i}$ a valid rectangular matrix.

We will define $\tilde{\mathbf{x}}^{j,i}_s = \begin{bmatrix} (\mathbf{x}^{j,i}_s)^T & (\mathbf{x}^{j-1}_s)^T & \cdots & (\mathbf{x}^1_s)^T \end{bmatrix}^T$. We make analogous defintions for $\tilde{\mathbf{v}}^{j,i}_s$, $\tilde{\hat{\mathbf{x}}}^{j,i}_s$ and $\tilde{\Delta}^{j,i}_{\tau_z}$.

For a matrix, let the absolute value operation $|\cdot|$ be applied component-wise. Let N denote the nilpotent matrix with ones on the super-diagonal and all other entries zero of appropriate dimension. Let $\mathbf{1}$ be the column vector with all entries 1 of appropriate dimension. Note that

$$\begin{aligned}
&|\tilde{\mathbf{A}}^R_{j,i}(\tilde{\mathbf{x}}^{j,i}_{\tau_z} + \tilde{\mathbf{v}}^{j,i}_{\tau_z} - \tilde{\hat{\mathbf{x}}}^{j,i}_{\tau_z})| \le |\tilde{\mathbf{A}}^R_{j,i}||\tilde{\mathbf{x}}^{j,i}_{\tau_z} + \tilde{\mathbf{v}}^{j,i}_{\tau_z} - \tilde{\hat{\mathbf{x}}}^{j,i}_{\tau_z}| \\
&\le \begin{bmatrix} |\mathbf{A}_{j,i}| & |\mathbf{M}_{j,i,j-1}| & \cdots & |\mathbf{M}_{j,i,1}| \end{bmatrix} \frac{1}{2}\tilde{\Delta}^{j,i}_{\tau_z} \\
&= \frac{1}{2}(|\lambda_{j,i}|\mathbf{I} + N)\Delta^{j,i}_{\tau_z} + \sum_{h=1}^{j-1} \frac{1}{2}|\mathbf{M}_{j,i,h}|\Delta^h_{\tau_z} \\
&\le \frac{1}{2}\Big(|\lambda_{j,i}| + \frac{\delta}{j}\Big)\Delta^{j,i}_{\tau_z} + \sum_{h=1}^{j-1} \frac{1}{2}|\mathbf{M}_{j,i,h}|\frac{\delta}{j n_h \kappa_{j,i,h}}\Delta^{j,i,d_{j,i}}_{\tau_z}\mathbf{1} && (4.11) \\
&\le \frac{1}{2}\Big(|\lambda_{j,i}| + \frac{\delta}{j}\Big)\Delta^{j,i}_{\tau_z} + \sum_{h=1}^{j-1} \frac{1}{2} n_h \kappa_{j,i,h}\frac{\delta}{j n_h \kappa_{j,i,h}}\Delta^{j,i,d_{j,i}}_{\tau_z}\mathbf{1} \\
&= \frac{1}{2}\Big(|\lambda_{j,i}| + \frac{\delta}{j}\Big)\Delta^{j,i}_{\tau_z} + \frac{1}{2}\frac{\delta(j-1)}{j}\Delta^{j,i,d_{j,i}}_{\tau_z}\mathbf{1} \\
&\le (|\lambda_{j,i}| + \delta)\frac{1}{2}\Delta^{j,i}_{\tau_z}, && (4.12)
\end{aligned}$$

where we have used the bin ordering (4.10) throughout and in particular for equation (4.11). Let $E_{j,i} = |\lambda_{j,i}| + \delta$. Let $\mathbf{E}_j = \mathrm{diag}(E_{j,1}\mathbf{I}, \ldots, E_{j,m_j}\mathbf{I})$ where each $\mathbf{I}$ is the identity matrix of appropriate size so that $E_{j,i}\mathbf{I} \in \mathbb{R}^{d_{j,i}\times d_{j,i}}$. Applying (4.12), we get

$$\begin{aligned}
&|\tilde{\mathbf{A}}^k_{j,i}(\tilde{\mathbf{x}}^{j,i}_{\tau_z} + \tilde{\mathbf{v}}^{j,i}_{\tau_z} - \tilde{\hat{\mathbf{x}}}^{j,i}_{\tau_z})| \le |\tilde{\mathbf{A}}^k_{j,i}||\tilde{\mathbf{x}}^{j,i}_{\tau_z} + \tilde{\mathbf{v}}^{j,i}_{\tau_z} - \tilde{\hat{\mathbf{x}}}^{j,i}_{\tau_z}| \\
&\le |\tilde{\mathbf{A}}^k_{j,i}|\frac{1}{2}\tilde{\Delta}^{j,i}_{\tau_z}
\end{aligned}$$

$$\leq |\tilde{\mathbf{A}}_{j,i}|^{k-1} \begin{bmatrix} (|\lambda_{j,i}| + \delta)\frac{1}{2}\Delta_{\tau_z}^{j,i} \\ \mathbf{E}_{j-1}\frac{1}{2}\Delta_{\tau_z}^{j-1} \\ \vdots \\ \mathbf{E}_1\frac{1}{2}\Delta_{\tau_z}^{1} \end{bmatrix}$$
$$\leq |\tilde{\mathbf{A}}_{j,i}|^{k-1}(|\lambda'_{j,i}| + \delta)\frac{1}{2}\tilde{\Delta}_{\tau_z}^{j,i}$$
$$\vdots$$
$$\leq (|\lambda'_{j,i}| + \delta)^k \frac{1}{2}\tilde{\Delta}_{\tau_z}^{j,i}. \tag{4.13}$$

Let us define

$$\mathbf{z}_{\tau_z,k} = |\mathbf{A}^k(\mathbf{x}_{\tau_z} + \mathbf{v}_{\tau_z} - \hat{\mathbf{x}}_{\tau_z})|.$$

From (4.13), we have

$$\mathbf{z}_{\tau_z,k}^{j,i} \leq (|\lambda'_{j,i}| + \delta)^k \frac{1}{2}\Delta_{\tau_z}^{j,i}, \quad \text{for all } 1 \leq j \leq M \text{ and } 1 \leq i \leq m_j, \tag{4.14}$$

where we apply our usual notational convention for vectors in $\mathbb{R}^n$.

Let $D_{j,i} = |\lambda'_{j,i}| + \delta$. Let $\mathbf{D}_j = \operatorname{diag}(D_{j,1}\mathbf{I}, \ldots, D_{j,m_j}\mathbf{I})$, where each $\mathbf{I}$ is the identity matrix of appropriate size so that $D_{j,i}\mathbf{I} \in \mathbb{R}^{d_{j,i}\times d_{j,i}}$. Let $\mathbf{D} = \operatorname{diag}(\mathbf{D}_M, \ldots, \mathbf{D}_1)$.

Then we can write (4.14) more compactly as

$$|\mathbf{A}^k(\mathbf{x}_{\tau_z} + \mathbf{v}_{\tau_z} - \hat{\mathbf{x}}_{\tau_z})| \leq \mathbf{D}^k \frac{1}{2}\Delta_{\tau_z}. \tag{4.15}$$

Now, consider the proof of Theorem 2.4.5. We wish to obtain a similar geometric bound on the difference between stopping times.

Let $\bar{K}_{j,i} = 1/(|\lambda'_{j,i}| + \delta + \epsilon - \eta)$. Let $\bar{\mathbf{K}}_j = \operatorname{diag}(\bar{K}_{j,1}\mathbf{I}, \ldots, \bar{K}_{j,m_j}\mathbf{I})$ where each $\mathbf{I}$ is the identity matrix of appropriate size so that $\bar{K}_{j,i}\mathbf{I} \in \mathbb{R}^{d_{j,i}\times d_{j,i}}$. Let $\bar{\mathbf{K}} = \operatorname{diag}(\bar{\mathbf{K}}_M, \ldots, \bar{\mathbf{K}}_1)$.

We put

$$\xi = \min_{j,i} K_{j,i}\bar{K}_{j,i} = \min_{j,i} \frac{\lceil |\lambda'_{j,i}| + \delta + \epsilon \rceil}{|\lambda'_{j,i}| + \delta + \epsilon - \eta} > 1.$$

Let us define the noise vector

$$\mathbf{w}_{\tau_z,k} = \mathbf{D}^{-k}\mathbf{A}^k\Big(-\mathbf{v}_{\tau_z} + \sum_{s=0}^{k-1} \mathbf{A}^{-1-s}\mathbf{w}_{\tau_z+s}\Big) + \mathbf{D}^{-k}\mathbf{v}_{\tau_z+k}. \tag{4.16}$$

It then follows that

$$\begin{aligned}
&P\left(|\mathbf{x}_{\tau_z+k} + \mathbf{v}_{\tau_z+k}| \nleq \frac{1}{2}\mathbf{K}\Delta_{\tau_z+k} \middle| \tau_{z+1} - \tau_z > k-1, \mathbf{x}_{\tau_z}, \Delta_{\tau_z} \right) \\
&= P\Bigg(\left| \mathbf{A}^k\mathbf{x}_{\tau_z} + \mathbf{A}^{k-1}\mathbf{u}_{\tau_z} + \sum_{s=0}^{k-1} \mathbf{A}^{k-1-s}\mathbf{w}_{\tau_z+s} + \mathbf{v}_{\tau_z+k} \right| \\
&\qquad\qquad \nleq \frac{1}{2}\mathbf{K}\Delta_{\tau_z+k} \Bigg| \tau_{z+1} - \tau_z > k-1, \mathbf{x}_{\tau_z}, \Delta_{\tau_z} \Bigg) \\
&= P\Bigg(\left| \mathbf{A}^k(\mathbf{x}_{\tau_z} - \hat{\mathbf{x}}_{\tau_z} + \mathbf{v}_{\tau_z} - \mathbf{v}_{\tau_z}) + \sum_{s=0}^{k-1} \mathbf{A}^{k-1-s}\mathbf{w}_{\tau_z+s} + \mathbf{v}_{\tau_z+k} \right| \\
&\qquad\qquad \nleq \frac{1}{2}\mathbf{K}\Delta_{\tau_z+k} \Bigg| \tau_{z+1} - \tau_z > k-1, \mathbf{x}_{\tau_z}, \Delta_{\tau_z} \Bigg) \\
&\leq P\Bigg(|\mathbf{A}^k(\mathbf{x}_{\tau_z} + \mathbf{v}_{\tau_z} - \hat{\mathbf{x}}_{\tau_z})| + \left| \mathbf{A}^k \left(-\mathbf{v}_{\tau_z} + \sum_{s=0}^{k-1} \mathbf{A}^{-1-s}\mathbf{w}_{\tau_z+s} \right) + \mathbf{v}_{\tau_z+k} \right| \\
&\qquad\qquad \nleq \frac{1}{2}\mathbf{K}\rho^{k-1}\mathbf{D}^{k-1}\mathbf{D}\bar{\mathbf{K}}\Delta_{\tau_z} \Bigg| \mathbf{x}_{\tau_z}, \Delta_{\tau_z} \Bigg) \\
&\leq P\left(\mathbf{D}^k\frac{1}{2}\Delta_{\tau_z} + \mathbf{D}^k|\mathbf{w}_{\tau_z,k}| \nleq \mathbf{D}^k\frac{1}{2}\rho^{k-1}\mathbf{K}\bar{\mathbf{K}}\Delta_{\tau_z} \middle| \mathbf{x}_{\tau_z}, \Delta_{\tau_z} \right) \\
&\leq P\left(|\mathbf{w}_{\tau_z,k}| \nleq \frac{1}{2}(\rho^{k-1}\xi - 1)\Delta_{\tau_z} \middle| \mathbf{x}_{\tau_z}, \Delta_{\tau_z} \right),
\end{aligned}$$

where we have used (4.15) and the commutativity of diagonal matrices with each other.

As in the proof of Theorem 2.2.1, by choosing appropriate values of ρ, ϵ and η we can form a countable state space $\mathcal{S}$ for the bin sizes and ensure that the Markov process $(\mathbf{x}_s, \Delta_s)$ is irreducible and aperiodic on $\mathbb{R}^n \times \mathcal{S}$.

A brief review of the proof of Theorems 2.2.1 and 3.0.3 then shows that we obtain the result provided that we can bound $Tr((\mathbf{D}^{-k}\mathbf{A}^k)^T\mathbf{D}^{-k}\mathbf{A}^k)$ by a polynomial in k. This new term comes from the form of (4.16).

Let $\mathbf{G}, \mathbf{H} \in \mathbb{R}^{n\times n}$ and let $\mathbf{G}_{i,j}, \mathbf{H}_{i,j}$ denote the component at row i and column j for $\mathbf{G}$ and $\mathbf{H}$ respectively. In the following, we will write $\mathbf{G} \leq \mathbf{H}$ when $\mathbf{G}_{i,j} \leq \mathbf{H}_{i,j}$ for all i, j.

By reordering the vectors in our Jordan transformation, we can reorder the blocks arbitrarily. Thus, we can assume without loss of generality that $|\lambda_{j,i}| \leq |\lambda_{j,k}|$ for $i > k$. For convenience, we relabel the eigenvalues $\lambda_{M,1}, \ldots, \lambda_{1,m_1}$ as $\lambda_n, \ldots, \lambda_1$. Similarly, we relabel $|\lambda'_{M,1}|, \ldots, |\lambda'_{1,m_1}|$ as $|\lambda'_n|, \ldots, |\lambda'_1|$. Thus, we have:

$$\lambda_i \leq \lambda'_i, \quad \text{for all } 1 \leq i \leq n, \tag{4.17}$$

$$|\lambda'_1| \leq \cdots \leq |\lambda'_n|. \tag{4.18}$$

We define $\mathbf{F} = \mathrm{diag}(\lambda_n, \ldots, \lambda_1)$ and $\mathbf{F}' = \mathrm{diag} = (|\lambda'_n|, \ldots, |\lambda'_1|)$. Then we can write $\mathbf{A} = \mathbf{F} + \mathbf{M}$ where $\mathbf{M}$ is some upper triangular nilpotent matrix. Letting $\{m_{i,j}\}$, we define

$$\alpha = \max\{1, |m_{i,j}| : 1 \leq i \leq n, 1 \leq j \leq n\}.$$

We define the absolute value operation $|\cdot|$ component-wise for matrices. It then follows that

$$|\mathbf{A}| = |\mathbf{F}| + |\mathbf{M}| \leq \mathbf{F}' + \alpha\mathbf{N},$$

where $\mathbf{N}$ is the matrix with all entries on and below the diagonal zero and all entries above the diagonal equal to one. That is, $\mathbf{N}$ is the nilpotent upper triangular matrix where all nonzero entries are one. Let $\bar{\mathbf{A}} = \mathbf{F}' + \alpha\mathbf{N}$. Applying (4.18), it can easily be verified by induction that

$$\bar{\mathbf{A}}^k \leq \begin{bmatrix} |\lambda'_n|^k & \binom{k}{1}\alpha|\lambda'_n|^{k-1} & \binom{k+1}{2}\alpha^2|\lambda'_n|^{k-1} & \cdots & \binom{k+n-2}{n-1}\alpha^{n-1}|\lambda'_n|^{k-1} \\ & |\lambda'_{n-1}|^k & \binom{k}{1}\alpha|\lambda'_{n-1}|^{k-1} & \cdots & \binom{k+n-3}{n-2}\alpha^{n-2}|\lambda'_{n-1}|^{k-1} \\ & & |\lambda'_{n-2}|^k & \ddots & \vdots \\ & & & \ddots & \binom{k}{1}\alpha|\lambda'_2|^{k-1} \\ & & & & |\lambda'_1|^k \end{bmatrix}.$$

Let us define

$$\tilde{\mathbf{A}}^k = \begin{bmatrix} 1 & \binom{k}{1}\alpha & \binom{k+1}{2}\alpha^2 & \cdots & \binom{k+n-2}{n-1}\alpha^{n-1} \\ & 1 & \binom{k}{1}\alpha & \cdots & \binom{k+n-3}{n-2}\alpha^{n-2} \\ & & 1 & \ddots & \vdots \\ & & & \ddots & \binom{k}{1}\alpha \\ & & & & 1 \end{bmatrix}.$$

The above then shows that

$$|\mathbf{A}|^k \leq \bar{\mathbf{A}}^k \leq (\mathbf{F}')^k \tilde{\mathbf{A}}^k.$$

Finally, using (4.18), we obtain

$$\begin{aligned} &Tr((\mathbf{D}^{-k}\mathbf{A}^k)^T\mathbf{D}^{-k}\mathbf{A}^k) \leq Tr((\mathbf{D}^{-k}|\mathbf{A}|^k)^T\mathbf{D}^{-k}|\mathbf{A}|^k) \\ &\leq Tr((\mathbf{D}^{-k}(\mathbf{F}')^k\tilde{\mathbf{A}}^k)^T\mathbf{D}^{-k}(\mathbf{F}')^k\tilde{\mathbf{A}}^k) \leq Tr((\tilde{\mathbf{A}}^k)^T\tilde{\mathbf{A}}^k) \\ &= \sum_{j=0}^{n-1}\sum_{i=0}^{j}\binom{k+i-1}{i}^2\alpha^{2i} \leq n\sum_{i=0}^{n-1}\binom{k+i-1}{i}^2\alpha^{2i} \\ &\leq n\alpha^{2n}\sum_{i=0}^{n-1}(k+i-1)^{2i} \leq n^2\alpha^{2n}(k+n)^{2n}. \end{aligned}$$

Thus, the bound is polynomial in k and we are done. □

Proof of Theorem 4.3.4: The proof follows directly from that of Theorem 4.3.3. Since the eigenvalues are ordered in decreasing magnitude, we can maintain the ordering of the bin sizes given in (4.10) without increasing the rate. Specifically, in the Proof of Theorem 4.3.3 we see that $|\lambda'_{j,i}| = |\lambda_{j,i}|$ for all $1 \leq j \leq M$, $1 \leq i \leq m_j$.□

Chapter 5

Multi-Controller Systems

In this chapter, we consider single-sensor, multi-controller systems.

5.1 Problem Statement

Consider the system

$$\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \sum_{j=1}^{M} \mathbf{B}^j \mathbf{u}_t^j + \mathbf{w}_t, \quad \mathbf{y}_t = \mathbf{C}\mathbf{x}_t + \mathbf{v}_t, \tag{5.1}$$

where $\mathbf{x}_t \in \mathbb{R}^n$, $\mathbf{y}_t \in \mathbb{R}^p$ are the state and sensor observation at time $t \in \mathbb{N}$. The control action exerted by controller j at time t is denoted by $\mathbf{u}_t^j \in \mathbb{R}^{m_j}$. The matrices $\mathbf{A}$, $\mathbf{B}^j, \mathbf{C}$ and random vectors $\mathbf{w}_t, \mathbf{v}_t$ are of compatible size. The initial state, $\mathbf{x}_0$, is drawn from the $\mathcal{Z}$ distribution.

We require that $\{\mathbf{w}_t\}$ and $\{\mathbf{v}_t\}$ be sequences of i.i.d. random vectors drawn from a distribution $\mathcal{Z}$, with finite $2+\epsilon$ moments in each component for some $\epsilon > 0$, which admits a probability density that is positive on every open set. At time t, $\mathbf{w}_t$ and $\mathbf{v}_t$ are independent of each other and the state $\mathbf{x}_t$.

Assumption 5.1.1. *We require joint controllability and observability. That is, the pair* $(\mathbf{A}, [\mathbf{B}^1 \quad \mathbf{B}^2 \quad \cdots \quad \mathbf{B}^M])$ *is controllable but the individual pairs* $(\mathbf{A}, \mathbf{B}^j)$ *may not be controllable. The pair* $(\mathbf{C}, \mathbf{A})$ *is observable.*

The setup is depicted in Figure 5.1. The observations are made by a single sensor which sends information to a set of M controllers through finite capacity channels. At each time stage t, we allow the sensor to send an encoded value $q_t^j \in \{1, 2, \ldots, N_t^j\}$ to controller j for some $N_t^j \in \mathbb{N}$. No feedback value b_t is needed in the multi-controller case since the sensor quantizes the entire state and knows when all quantizers are perfectly-zoomed. We define the rate at time t as $R_t = \sum_{j=1}^{M} \log_2(N_t^j)$. The coding scheme is applied periodically with period T and so the rate for all time stages is specified by $\{N_0^j, \ldots, N_{T-1}^j : 1 \leq j \leq M\}$. The average rate is

$$R_{\text{avg}} = \frac{1}{T} \sum_{t=0}^{T-1} R_t. \tag{5.2}$$

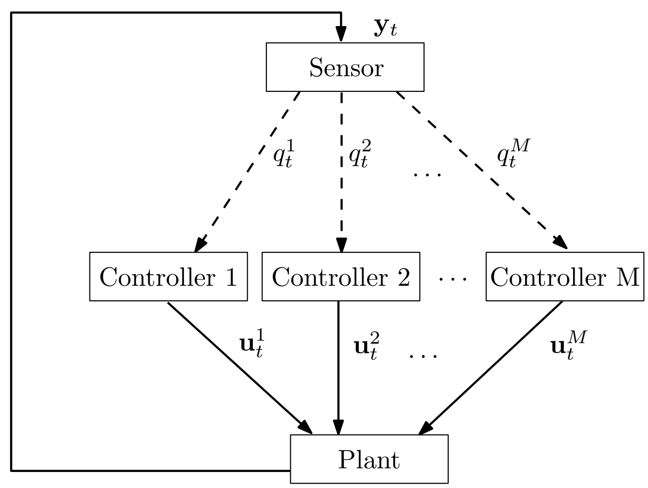

Figure 5.1: A multi-controller system with finite-rate communication channels.

Information structure. For a process $\{\mathbf{x}_t\}$ we define $\mathbf{x}_{[a,b]} = \{\mathbf{x}_a, \mathbf{x}_{a+1}, \ldots, \mathbf{x}_b\}$. At time t, the sensor maps its information

$$I_t^s := \mathbf{y}_{[0,t]} \to (q_t^1, \cdots, q_t^M) \in \prod_{j=1}^{M} \{1, \ldots, N_t^j\}.$$

Each controller j maps its information $I_t^{c_j} := q_{[0,t]}^j \to \mathbf{u}_t^j \in \mathbb{R}^{m_j}$.

The controllability matrix of controller j is

$$\mathcal{C}_{(\mathbf{A},\mathbf{B}^j)} = \begin{bmatrix} \mathbf{B}^j & \mathbf{A}\mathbf{B}^j & \cdots & \mathbf{A}^{n-1}\mathbf{B}^j \end{bmatrix}.$$

We denote the controllable subspace of controller j by K^j, which is the range space of $\mathcal{C}_{(\mathbf{A},\mathbf{B}^j)}$. The observability matrix of the sensor is

$$\mathcal{O}_{(\mathbf{C},\mathbf{A})} = \begin{bmatrix} (\mathbf{C})^T & (\mathbf{C}\mathbf{A})^T & \cdots & (\mathbf{C}\mathbf{A}^{n-1})^T \end{bmatrix}^T,$$

the null space is $N = \mathrm{Ker}(\mathcal{O}_{(\mathbf{C},\mathbf{A})})$ and the observable subspace is defined to be $O = N^{\perp}$.

5.2 Main Result

Let $V_1, \ldots, V_\ell$ denote the generalized eigenspaces of $\mathbf{A}$. We state the following assumption. In Section 5.3 we will remove this assumption and consider the general case.

Assumption 5.2.1. *Each eigenspace is controlled by some controller. That is, for each $1 \leq i \leq \ell$ there exists a $1 \leq j \leq M$ such that $V_i \subseteq K^j$.*

We label the eigenvalues of $\mathbf{A}$ as $\lambda_1, \ldots, \lambda_n$. Without loss, we assume that $|\lambda_i| > 1$ for all $1 \leq i \leq n$. Our main result for multi-controller systems is the following:

Theorem 5.2.2. *Under Assumption 5.2.1, there exists a coding and control policy with average rate $R_{avg} \leq 1/(T2n)(\sum_{i=1}^{n} \log_2(\lceil |\lambda_i|^{T2n} + \epsilon \rceil + 1))$ for some $\epsilon > 0$ which gives:*

(a) the existence of a unique invariant distribution for $\{\mathbf{x}_{2nt}\}$;

(b) $\lim_{t\to\infty} E[\|\mathbf{x}_{2nt}\|_2^2] < \infty$.

Theorem 5.2.3. *The average rate in Theorem 5.2.2 achieves the minimum rate* (1.9) *asymptotically for large sampling periods. That is,* $\lim_{T\to\infty} R_{avg} = R_{\min}$.

Proof of Theorem 5.2.3: Follows from the proof of Theorem 2.2.2. □

The proof of Theorem 5.2.2 is basically an application of the Jordan normal form together with Assumption 5.2.1.

Proof of Theorem 5.2.2: The proof follows that of Theorem 4.2.2. Instead of working with left eigenvectors, however, we now work with right eigenvectors. Under Assumption 5.2.1, we can assign each eigenspace $V_i \subseteq K^j$ to some controller j. Let $V_{j,1}, \ldots, V_{j,m_j}$ denote the eigenspaces assigned to controller j and let us write $V_{j,i} = \text{span}\{\mathbf{v}_{j,i,1}, \ldots, \mathbf{v}_{j,i,d_{j,i}}\}$ where each $\mathbf{v}_{j,i,h} \in \mathbb{R}^{n\times 1}$ is a generalized eigenvector. We put

$$\begin{aligned}\mathbf{Q}_{j,i} &= \begin{bmatrix}\mathbf{v}_{j,i,1} & \cdots & \mathbf{v}_{j,i,d_{j,i}}\end{bmatrix},\\ \mathbf{Q}_j &= \begin{bmatrix}\mathbf{Q}_{j,1} & \cdots & \mathbf{Q}_{j,m_j}\end{bmatrix}\\ \mathbf{Q} &= \begin{bmatrix}\mathbf{Q}_M & \cdots & \mathbf{Q}_1\end{bmatrix}.\end{aligned}$$

Consider the first n time stages of our control policy. The basic recursion for LTI systems applied to (5.1) yields

$$\mathbf{x}_n = \mathbf{A}^n\mathbf{x}_0 + \sum_{j=1}^{M}\sum_{t=0}^{n-1}\mathbf{A}^{n-1-t}\mathbf{B}^j\mathbf{u}_t^j + \sum_{t=0}^{n-1}\mathbf{A}^{n-1-t}\mathbf{w}_t.$$

By our controllability assumption, it follows that we can choose $\mathbf{u}_0^j, \ldots, \mathbf{u}_{n-1}^j$ such that

$$\sum_{t=0}^{n-1}\mathbf{A}^{n-1-t}\mathbf{B}^j\mathbf{u}_t^j = \sum_{i=1}^{m_j}\sum_{h=1}^{d_{j,i}}\bar{u}_0^{j,i,h}\mathbf{v}_{j,i,h},$$

where each $\bar{u}_0^{j,i,h} \in \mathbb{R}$ is chosen arbitrarily by controller j. Now, let us define

$$\begin{aligned}\bar{\mathbf{u}}_0^{j,i} &= \begin{bmatrix}\bar{u}_0^{j,i,1} & \cdots & \bar{u}_0^{j,i,d_{j,i}}\end{bmatrix}^T,\\ \bar{\mathbf{u}}_0^{j} &= \begin{bmatrix}(\bar{\mathbf{u}}_0^{j,1})^T & \cdots & (\bar{\mathbf{u}}_0^{j,m_j})^T\end{bmatrix}^T,\\ \bar{\mathbf{u}}_0 &= \begin{bmatrix}(\bar{\mathbf{u}}_0^{M})^T & \cdots & (\bar{\mathbf{u}}_0^{1})^T\end{bmatrix}^T.\end{aligned}$$

Our recursion then becomes

$$\mathbf{x}_n = \mathbf{A}^n \mathbf{x}_0 + \mathbf{Q}\bar{\mathbf{u}}_0 + \sum_{t=0}^{n-1} \mathbf{A}^{n-1-t}\mathbf{w}_t.$$

We apply the similarity transform $\bar{\mathbf{x}}_t = \mathbf{Q}^{-1}\mathbf{x}_t$ and define $\bar{\mathbf{A}} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}$ and $\bar{\mathbf{w}}_0 = \mathbf{Q}^{-1}\sum_{t=0}^{n-1} \mathbf{A}^{n-1-t}\mathbf{w}_t$. Our recursion is thus

$$\bar{\mathbf{x}}_n = \bar{\mathbf{A}}^n \bar{\mathbf{x}}_0 + \bar{\mathbf{u}}_0 + \bar{\mathbf{w}}_0.$$

In the above, we apply our control action over the first n time stages. Suppose that instead we allow the sensor to make observations in the first n time stages and apply a control in the second n time stages, as in the proof of Theorem 2.2.1. Then we apply our control policy repeatedly, every $2n$ time stages. By a slight abuse of notation, we can define $\mathbf{x}_s = \bar{\mathbf{x}}_{2ns}$, $\mathbf{u}_s = \bar{\mathbf{u}}_{2ns}$, $\mathbf{w}_s = \bar{\mathbf{w}}_{2ns}$ to get the system

$$\mathbf{x}_{s+1} = \bar{\mathbf{A}}^{2n}\mathbf{x}_s + \mathbf{u}_s + \mathbf{w}_s, \qquad \mathbf{y}_s = \mathbf{x}_s + \mathbf{v}_s$$

The vector $\mathbf{y}_s$ is known by the sensor at time s through periodic observations and $\{\mathbf{w}_s\}$, $\{\mathbf{v}_s\}$ are i.i.d. sequences of random vectors with $2+\epsilon$ moments for some $\epsilon > 0$.

Furthermore, since $\mathbf{Q}$ is the Jordan normal transformation matrix, it follows that $\bar{\mathbf{A}}^{2n} = \mathrm{diag}(\bar{\mathbf{J}}_1^{2n}, \ldots, \bar{\mathbf{J}}_\ell^{2n})$ where each $\bar{\mathbf{J}}_i \in \mathbb{R}^{d_i \times d_i}$ is a Jordan block. Since we can apply another Jordan transformation to this sampled system, we can assume without loss of generality that $\bar{\mathbf{A}}^{2n}$ is actually in Jordan form and each $\bar{\mathbf{J}}_i^{2n}$ is actually a Jordan block.

To simplify notation, we write $\mathbf{A} := \bar{\mathbf{A}}^{2n} = \mathrm{diag}(\mathbf{J}_1, \ldots, \mathbf{J}_\ell)$ where each $\mathbf{J}_i \in \mathbb{R}^{d_i \times d_i}$ is a Jordan block, $\mathbf{x}_s = \begin{bmatrix}(\mathbf{x}_s^1)^T & \cdots & (\mathbf{x}_s^\ell)^T\end{bmatrix}^T$ where $\mathbf{x}_s^i \in \mathbb{R}^{d_i}$ and similarly for $\mathbf{u}_s, \mathbf{w}_s, \mathbf{y}_s$ and $\mathbf{v}_s$. From the above, we can also see that, for each i, there exists a j such that $\mathbf{u}_s^i$ is determined arbitrarily by controller j at time s.

Thus our system is equivalent to the following subsystems:

$$\mathbf{x}_{s+1}^i = \mathbf{J}_i \mathbf{x}_s^i + \mathbf{u}_s^i + \mathbf{w}_s^i, \qquad \mathbf{y}_s^i = \mathbf{x}_s^i + \mathbf{v}_s^i \qquad 1 \le i \le \ell$$

where for each $1 \le i \le \ell$, there exists a controller j which can choose $\mathbf{u}_s^i$ arbitrarily.

We apply a policy similar to that in Section 2.3. At each time stage s, the sensor sends an estimate of $\mathbf{y}_s^i$ encoded in accordance with our modified uniform quantizer and sent to controller j through q_t^j. The problem is thus reduced to the single-station case and we are done. □

5.3 Sufficient Conditions for the General Multi-Controller Case

The following theorem extends the classical controllability canonical decomposition to the decentralized case. For a detailed proof in the centralized case, see [24]. The more general multi-agent setup, where each agent makes observations and applies a control action, can be found in [15].

Theorem 5.3.1. *Under Assumption 5.1.1, there exists a matrix* $\mathbf{Q}$ *such that, if we define* $\bar{\mathbf{A}} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}$ *and* $\bar{\mathbf{B}}^j = \mathbf{Q}^{-1}\mathbf{B}^j$*, then*

$$\bar{\mathbf{A}} = \begin{bmatrix} \bar{\mathbf{A}}_1 & * & \cdots & * \\ & \bar{\mathbf{A}}_2 & \cdots & * \\ & & \ddots & \\ & 0 & & \bar{\mathbf{A}}_M \end{bmatrix}, \tag{5.3a}$$

$$\begin{bmatrix} \bar{\mathbf{B}}^1 & \bar{\mathbf{B}}^2 & \cdots & \bar{\mathbf{B}}^M \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{B}}_C^1 & * & \cdots & * \\ & \bar{\mathbf{B}}_C^2 & \cdots & * \\ & & \ddots & \\ & 0 & & \bar{\mathbf{B}}_C^M \end{bmatrix}, \tag{5.3b}$$

where the $*$*'s denote irrelevant submatrices, each* $\bar{\mathbf{A}}_j \in \mathbb{R}^{n_j \times n_j}$ *and each* $\bar{\mathbf{B}}_C^j \in \mathbb{R}^{n_j \times m_j}$*.*

Proof of Theorem 5.3.1: The proof follows that of Theorem 4.3.1 from the multi-sensor case. We define $n_1 = \dim(K^1)$, and $n_j = \dim(K^j) - \dim(K^j \cap (\cup_{i=1}^{j-1} K^i))$, for $2 \le j \le M$. We choose n_1 linearly independent column vectors from $\mathcal{C}_{(\mathbf{A},\mathbf{B}^1)}$ and label them $\mathbf{q}_1^1, \ldots, \mathbf{q}_{n_1}^1$. Proceeding by induction, we choose $\{\mathbf{q}_1^j, \ldots, \mathbf{q}_{n_j}^j\}$ from $\mathcal{C}_{(\mathbf{A},\mathbf{B}^j)}$ such that

$$\{\mathbf{q}_1^1, \ldots, \mathbf{q}_{n_1}^1, \mathbf{q}_1^2, \ldots, \mathbf{q}_{n_2}^2, \ldots, \mathbf{q}_1^j, \ldots, \mathbf{q}_{n_j}^j\}$$

is a set of linearly independent vectors.

We define $\mathbf{Q}^j = \begin{bmatrix} \mathbf{q}_1^j & \cdots & \mathbf{q}_{n_j}^j \end{bmatrix}$ for all $1 \leq j \leq M$ and concatenate these matrices to choose our transformation matrix $\mathbf{Q} = \begin{bmatrix} \mathbf{Q}^1 & \cdots & \mathbf{Q}^M \end{bmatrix}$. It will also be convenient to denote the columns of $\mathbf{Q}$ by $\mathbf{q}_1, \ldots, \mathbf{q}_n$ so that $\mathbf{Q} = \begin{bmatrix} \mathbf{q}_1 & \cdots & \mathbf{q}_n \end{bmatrix}$.

From the Cayley–Hamilton Theorem, we know that for all $m \geq n$ there exist $\alpha_0, \ldots, \alpha_{n-1}$ such that $\mathbf{A}^m = \sum_{i=0}^{n-1} \alpha_i \mathbf{A}^i$. Since $\{\mathbf{q}_1^j, \ldots, \mathbf{q}_{n_j}^j\}$ are columns of $\mathcal{C}_{(\mathbf{A},\mathbf{B}^j)}$, this implies that $\mathbf{A}\mathbf{q}_i^j$ is in the column space of $\mathcal{C}_{(\mathbf{A},\mathbf{B}^j)}$ for all $1 \leq i \leq n_j$. Let us define the sets

$$S_j := \{\mathbf{q}_1^1, \ldots, \mathbf{q}_{n_1}^1, \mathbf{q}_1^2, \ldots, \mathbf{q}_{n_2}^2, \ldots, \mathbf{q}_1^j, \ldots, \mathbf{q}_{n_j}^j\}, \qquad 1 \leq j \leq M$$

From our construction, it is then clear that

$$\mathbf{A}\mathbf{q}_1^j, \ldots, \mathbf{A}\mathbf{q}_{n_j}^j \in \operatorname{span}(S_j).$$

Our similarity transform gives

$$\mathbf{Q}\bar{\mathbf{A}} = \mathbf{A}\mathbf{Q}. \tag{5.4}$$

We write $\bar{\mathbf{A}}$ in terms of its column vectors as $\bar{\mathbf{A}} = \begin{bmatrix} \bar{\mathbf{a}}_1 & \cdots & \bar{\mathbf{a}}_n \end{bmatrix}$ with $\bar{\mathbf{a}}_i = \begin{bmatrix} \bar{a}_{i,1} & \cdots & \bar{a}_{i,n} \end{bmatrix}^T$ and $\bar{a}_{i,h} \in \mathbb{R}$. If we write $\mathbf{A}\mathbf{Q} = \begin{bmatrix} \mathbf{A}\mathbf{q}_1 & \cdots & \mathbf{A}\mathbf{q}_n \end{bmatrix}$, then the relation (5.4) becomes

$$\sum_{h=1}^{n} \bar{a}_{i,h}\mathbf{q}_h = \mathbf{A}\mathbf{q}_i$$

for $1 \leq i \leq n$ and we can see that the i^{th} column of $\bar{\mathbf{A}}$ is the representation of $\mathbf{A}\mathbf{q}_i$ with repect to the basis $\mathbf{q}_1, \ldots, \mathbf{q}_n$. This, coupled with the observation that $\mathbf{Q}\bar{\mathbf{B}}^j = \mathbf{B}^j$ and $\mathbf{B}^j$ is in the column space of $\mathcal{C}_{(\mathbf{A},\mathbf{B}^j)}$ gives the desired form. □

Remark 5.3.2. *In the proof of Theorem 5.3.1, we give one construction for the triangular decomposition in* (5.3)*. This transformation is not unique. There may be many ways to achieve a block upper triangular form and it is not necessary to place the controllers in order* $1, \ldots, M$*.*

Theorem 5.3.3. *There exists a coding and control policy which gives:*

(a) the existence of a unique invariant distribution for $\{\mathbf{x}_{2nt}\}$*;*

(b) $\lim_{t\to\infty} E[\|\mathbf{x}_{2nt}\|_2^2] < \infty$*,*

and with average rate in the limit of large sampling periods

$$\lim_{T\to\infty} R_{avg} = \sum_{j=1}^{M}\sum_{i=1}^{n_j} \log_2(\max\{|\lambda_{j,i}|, |\lambda_{h,\ell}| : h > j, 1 \le \ell \le n_h\}).$$

Clearly, we could also achieve (a) and (b) in Theorem 4.3.3 with $\lim_{T\to\infty} R_{\text{avg}} = n\log_2(\lambda_{\text{absmax}})$ where $\lambda_{\text{absmax}} = \max_{j,i}\{|\lambda_{j,i}|\}$.

For Theorem 5.3.4 below, recall that we have some flexibility in the decomposition given by Theorem 5.3.1. See the proof of Theorem 5.3.1 and Remark 5.3.2.

Theorem 5.3.4. *If the eigenvalues of* $\bar{\mathbf{A}}_1, \ldots, \bar{\mathbf{A}}_M$ *in* (4.4) *are ordered in decreasing magnitude then Theorem 5.2.2 holds without Assumption 5.2.1. That is, the theorem holds if for* $\lambda_i \in \Lambda(\bar{\mathbf{A}}_i)$ *and* $\lambda_j \in \Lambda(\bar{\mathbf{A}}_j)$ *we have that* $|\lambda_i| \ge |\lambda_j|$ *when* $i < j$*.*

5.4 Coding and Control Policy for the General Multi-Controller Case

Consider the system (5.1). Sampling, observing and controlling as in the proof of Theorem 2.2.1 and applying the transform $\bar{\mathbf{x}}_t = \mathbf{Q}^{-1}\mathbf{x}_t$ where $\mathbf{Q}^{-1}$ is given in Theorem 5.3.1, we obtain the system

$$\mathbf{x}_{s+1} = \mathbf{A}\mathbf{x}_s + \mathbf{u}_s + \mathbf{w}_s, \qquad \mathbf{y}_s = \mathbf{x}_s + \mathbf{v}_s.$$

We do not relabel the variables (for example $\bar{\mathbf{x}}_s$) by a slight abuse of notation and for the sake of reasonable presentation.

In the above, $\mathbf{A}$ is block upper triangular with the blocks $\mathbf{A}_M, \ldots, \mathbf{A}_1$ descending along the diagonal and each $\mathbf{A}_j \in \mathbb{R}^{n_j\times n_j}$ as in (4.4a) of Theorem 5.3.1. Recall Remark 5.3.2 and that we are free to reorder the block so that $\mathbf{A}_M$ appears leftmost in $\mathbf{A}$ and $\mathbf{A}_1$ appears rightmost. We simply redefine $n_M = \dim(K^M)$, and $n_j =$

$\dim(K^j) - \dim(K^j \cap (\cup_{i=j+1}^{M} K^i))$, for $1 \leq j \leq M-1$ and proceed with the proof of Theorem 5.3.1 accordingly.

Since we can always apply a block transformation to $\mathbf{A}$ in which each of the blocks is the Jordan tranformation of $\mathbf{A}_j$, we can assume without loss of generality that each $\mathbf{A}_j$ is in real Jordan normal form and we write $\mathbf{A}_j = \text{diag}(\mathbf{A}_{j,1}, \ldots, \mathbf{A}_{j,m_j})$ where each $\mathbf{A}_{j,i} \in \mathbb{R}^{d_{j,i} \times d_{j,i}}$ is a Jordan block.

Let us write

$$\mathbf{x}_s = \begin{bmatrix} (\mathbf{x}_s^M)^T & \cdots & (\mathbf{x}_s^1)^T \end{bmatrix}^T$$

where

$$\mathbf{x}_s^j = \begin{bmatrix} (\mathbf{x}_s^{j,1})^T & \cdots & (\mathbf{x}_s^{j,m_j})^T \end{bmatrix}^T$$

and $\mathbf{x}_s^j \in \mathbb{R}^{n_j}$ with

$$\mathbf{x}_s^{j,i} = \begin{bmatrix} x_s^{j,i,1} & \cdots & x_s^{j,i,d_{j,i}} \end{bmatrix}^T$$

and each $x_s^{j,i,h} \in \mathbb{R}$. We will use the same notational convention for all relevant vectors in this section. Namely, we will follow this convention for $\mathbf{u}_s$, $\mathbf{w}_s$, $\mathbf{y}_s$, $\mathbf{v}_s$ and for Δ_s, $\hat{\mathbf{x}}_s$, $\mathbf{L}$ which will be specified.

From the proof of Theorem 5.3.1, we know that the columns of $\mathbf{Q}$ are taken from the column spaces of $\{\mathcal{C}_{(\mathbf{A},\mathbf{B}^j)}\}$. By applying the transform $\mathbf{Q}^{-1}$, we can see by the usual recursion and the identity $\mathbf{Q}^{-1}\mathbf{Q} = \mathbf{I}$ that each controller j can choose $\mathbf{u}_s^j$ arbitrarily.

Let us denote the eigenvalue of $\mathbf{A}_{j,i}$ by $\lambda_{j,i}$. We define

$$|\lambda'_{j,i}| = \max\{|\lambda_{j,i}|, |\lambda_{h,\ell}| : 1 \leq h < j, 1 \leq \ell \leq m_h\}.$$

Let $K_{j,i} = \lceil |\lambda'_{j,i}| + \delta + \epsilon \rceil$ for some $\delta, \epsilon > 0$. Let $\mathbf{K}_j = \text{diag}(K_{j,1}\mathbf{I}, \ldots, K_{j,m_j}\mathbf{I})$ where each $\mathbf{I}$ is the identity matrix of appropriate size so that $K_{j,i}\mathbf{I} \in \mathbb{R}^{d_{j,i} \times d_{j,i}}$. Let $\mathbf{K} = \text{diag}(\mathbf{K}_M, \ldots, \mathbf{K}_1)$.

Let $\Delta_s^{j,i,h}$ be the bin size corresponding to the component $x_s^{j,i,h}$ at time s. We let $q_s^{j,i,h} = \mathcal{E}_{K_{j,i}}^{\Delta_s^{j,i,h}}(y_s^{j,i,h})$. Let our fixed rate for sensor j be $N^j = (\prod_{i=1}^{m_j} K_{j,i}^{d_{j,i}}) + 1$ for all $s \in \mathbb{N}$. Choose any invertible function

$$f_j : \prod_{i=1}^{m_j} \{1, \ldots, K_{j,i}\}^{d_{j,i}} \to \left\{1, \ldots, \prod_{i=1}^{m_j} K_{j,i}^{d_{j,i}}\right\}.$$

We then choose the encoded value

$$q_s^j = \begin{cases} f_j(q_s^{j,1,1}, \dots, q_s^{j,m_j,d_{j,m_j}}), & \text{if } q_s^{j,i,h} \neq 0 \text{ for all } 1 \leq j \leq M,\ 1 \leq i \leq m_j,\ 1 \leq h \leq d_{j,i}, \\ 0, & \text{otherwise.} \end{cases}$$

Upon receiving $q_s^j \neq 0$, controller j knows

$$\{q_s^{j,i,h} : 1 \leq i \leq m_j, 1 \leq h \leq d_{j,i}\}.$$

Each controller j forms the estimate $\hat{\mathbf{x}}_s^j$ as

$$\hat{\mathbf{x}}_s^j = \begin{bmatrix} (\hat{\mathbf{x}}_s^{j,1})^T & \cdots & (\hat{\mathbf{x}}_s^{j,m_j})^T \end{bmatrix}^T,$$

where

$$\hat{\mathbf{x}}_s^{j,i} = \begin{bmatrix} \hat{x}_s^{j,i,1} & \cdots & \hat{x}_s^{j,i,d_{j,i}} \end{bmatrix}^T$$

and

$$\hat{x}_s^{j,i,h} = \begin{cases} \mathcal{D}_{K_{j,i}}^{\Delta_s^{j,i,h}}(q_s^{j,i,h}), & \text{if } q_s^j \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Note that we do not need the feedback bit b_s in the multi-controller case since the sensor recovers $\mathbf{x}_s + \mathbf{v}_s$ and chooses $q_s^j \neq 0$ only when all quantizers are perfectly zoomed. Thus, when $q_s^j \neq 0$, controller j knows that $s = \tau_z$ for some $z \in \mathbb{N}$.

The update equations are

$$\Delta_{s+1}^{j,i} = \bar{Q}^{j,i}\left(q_s^j, \Delta_s^{j,i}\right)\Delta_s^{j,i}, \quad \bar{Q}^{j,i}\left(q_s^j, \Delta_s^{j,i}\right) = \begin{cases} \rho(|\lambda'_{j,i}| + \delta), & \text{if } q_s^j = 0, \\ \beta_{j,i}(\Delta_s^{j,i}), & \text{otherwise,} \end{cases} \tag{5.5}$$

for some $\rho > 1$ and with

$$\beta_{j,i}(\Delta_s^{j,i}) = \operatorname{diag}(\beta_{j,i,1}(\Delta_s^{j,i,1}), \dots, \beta_{j,i,d_{j,i}}(\Delta_s^{j,i,d_{j,i}})), \tag{5.6a}$$

$$\beta_{j,i,h}(\Delta_s^{j,i,h}) = \begin{cases} 1, & \text{if } \Delta_s^{j,i,h} \leq L^{j,i,h}, \\ \frac{|\lambda'_{j,i}|+\delta}{|\lambda'_{j,i}|+\delta+\epsilon-\eta}, & \text{otherwise,} \end{cases} \tag{5.6b}$$

for some $0 < \eta < \epsilon$ and $\mathbf{L}^{j,i} \in \mathbb{R}_+^{d_{j,i}}$. Note that if we define $\bar{\mathbf{L}}^{j,i} = \mathbf{L}^{j,i}(|\lambda'_{j,i}|+\delta)/(|\lambda'_{j,i}|+\delta+\epsilon-\eta)$ then $\Delta_s > \bar{\mathbf{L}}$ for all $s \in \mathbb{N}$.

Bin ordering. From (5.3a) of Theorem 5.3.1, we can write

$$\mathbf{A} = \begin{bmatrix}(\mathbf{A}_M^R)^T & \cdots & (\mathbf{A}_1^R)^T\end{bmatrix}^T$$

where each $\mathbf{A}_j^R \in \mathbb{R}^{n_j \times n}$. We can further write $\mathbf{A}_j^R = \begin{bmatrix}0 & \mathbf{A}_j & \mathbf{M}_j\end{bmatrix}$ where

$$\mathbf{A}_j = \operatorname{diag}(\mathbf{A}_{j,1}, \ldots, \mathbf{A}_{j,m_j})$$

and

$$\mathbf{M}_j = \begin{bmatrix}(\mathbf{M}_{j,1})^T & \cdots & (\mathbf{M}_{j,m_j})^T\end{bmatrix}^T$$

with

$$\mathbf{M}_{j,i} = \begin{bmatrix}\mathbf{M}_{j,i,j-1} & \cdots & \mathbf{M}_{j,i,1}\end{bmatrix}$$

and each $\mathbf{M}_{j,i,h} \in \mathbb{R}^{d_{j,i},n_h}$.

Let us denote the entries of $\mathbf{M}_{j,i,h}$ by $\{m_{j,i,h}^{k,\ell}\}$. We define the entry of maximum absolute value as

$$\kappa_{j,i,h} = \max_{k,\ell}\{|m_{j,i,h}^{k,\ell}|\}.$$

We set $\mathbf{L} = c\Delta_0$, for some $0 < c \leq 1$. For any $\delta > 0$, by our coding and control policy (and in particular the choice of $\{|\lambda'_{j,i}|\}$) given above, we can choose Δ_0 such that the following ordering is maintained for all $s \in \mathbb{N}$:

$$\Delta_s^{j,i,h+1} \leq \frac{\delta}{j}\Delta_s^{j,i,h}, \tag{5.7a}$$
$$1 \leq j \leq M,\ 1 \leq i \leq m_j,\ 1 \leq h \leq d_{j,i}-1,$$
$$\Delta_s^{j,k,1} \leq \Delta_s^{j,i,1}, \tag{5.7b}$$
$$1 \leq j \leq M,\ 1 \leq i \leq m_j - 1,\ i < k \leq m_j,$$
$$\Delta_s^{k,\ell,h} \leq \frac{\delta}{j n_k \kappa_{j,i,k} K_{k,\ell}}\Delta_s^{j,i,d_{j,i}}, \tag{5.7c}$$
$$2 \leq j \leq M,\ 1 \leq i \leq m_j,\ 1 \leq k \leq j-1,\ 1 \leq \ell \leq m_k,\ 1 \leq h \leq d_{k,\ell},$$

Informally, we order the bins within Jordan blocks $\mathbf{A}_{j,i}$, within controller blocks

$\mathbf{A}_j$ and between controller blocks $\mathbf{A}_j$. Note that in (5.7) there is an extra $K_{k,\ell} = \lceil |\lambda_{k,\ell}| + \delta + \epsilon \rceil$ term and thus (5.7) implies (4.10).

Control action. We choose the control actions

$$\mathbf{u}_s^j = -\mathbf{A}_j \hat{\mathbf{x}}_s^j,$$

leading to the joint action

$$\mathbf{u}_s = -\mathrm{diag}(\mathbf{A}_M, \ldots, \mathbf{A}_1)\hat{\mathbf{x}}_s.$$

We define the sequence of stopping times

$$\tau_0 = 0, \quad \tau_{z+1} = \min\left\{ s > \tau_z : |\mathbf{y}_s| = |\mathbf{x}_s + \mathbf{v}_s| \leq \frac{1}{2}\mathbf{K}\Delta_s \right\}.$$

Proof of Theorems 5.3.3 and 5.3.4: Let $\lambda \in \mathbb{R}$. The proof is similar for $\lambda \in \mathbb{C}$. Let us define $\bar{\mathbf{A}} = \mathrm{diag}(\mathbf{A}_M, \ldots, \mathbf{A}_1)$ and write $\mathbf{A} = \bar{\mathbf{A}} + \mathbf{N}$ where $\mathbf{N}$ is the correct nilpotent matrix.

Let $D_{j,i} = |\lambda'_{j,i}| + \delta$. Let $\mathbf{D}_j = \mathrm{diag}(D_{j,1}\mathbf{I}, \ldots, D_{j,m_j}\mathbf{I})$ where each $\mathbf{I}$ is the identity matrix of appropriate size so that $D_{j,i}\mathbf{I} \in \mathbb{R}^{d_{j,i} \times d_{j,i}}$. Let $\mathbf{D} = \mathrm{diag}(\mathbf{D}_M, \ldots, \mathbf{D}_1)$.

Let us define the noise process

$$\mathbf{w}_{\tau_z,k} = \mathbf{D}^{-k}\mathbf{A}^k\Big(-\mathbf{v}_{\tau_z} + \sum_{s=0}^{k-1} \mathbf{A}^{-1-s}\mathbf{w}_{\tau_z+s}\Big) + \mathbf{D}^{-k}\mathbf{v}_{\tau_z+k}.$$

For a matrix, we define the absolute value operation $|\cdot|$ component-wise. It then follows that

$$\begin{aligned}
|\mathbf{x}_{\tau_z+k} + \mathbf{v}_{\tau_z+k}| &= \left| \mathbf{A}^k \mathbf{x}_{\tau_z} + \mathbf{A}^{k-1}\mathbf{u}_{\tau_z} + \sum_{s=0}^{k-1} \mathbf{A}^{k-1-s}\mathbf{w}_{\tau_z+s} + \mathbf{v}_{\tau_z+k} \right| \\
&= \left| \mathbf{A}^{k-1}(\mathbf{A}(\mathbf{x}_{\tau_z} + \mathbf{v}_{\tau_z} - \mathbf{v}_{\tau_z}) - \bar{\mathbf{A}}\hat{\mathbf{x}}_{\tau_z}) + \sum_{s=0}^{k-1} \mathbf{A}^{k-1-s}\mathbf{w}_{\tau_z+s} + \mathbf{v}_{\tau_z+k} \right| \\
&= \left| \mathbf{A}^{k-1}((\bar{\mathbf{A}} + \mathbf{N})(\mathbf{x}_{\tau_z} + \mathbf{v}_{\tau_z}) - \bar{\mathbf{A}}\hat{\mathbf{x}}_{\tau_z}) + \mathbf{A}^k\Big(-\mathbf{v}_{\tau_z} + \sum_{s=0}^{k-1} \mathbf{A}^{-1-s}\mathbf{w}_{\tau_z+s}\Big) + \mathbf{v}_{\tau_z+k} \right|
\end{aligned}$$

$$
\begin{aligned}
&= \left|\mathbf{A}^{k-1}(\bar{\mathbf{A}}(\mathbf{x}_{\tau_z} + \mathbf{v}_{\tau_z} - \hat{\mathbf{x}}_{\tau_z}) + \mathbf{N}(\mathbf{x}_{\tau_z} + \mathbf{v}_{\tau_z})) + \mathbf{D}^k \mathbf{w}_{\tau_z,k}\right| \\
&\leq |\mathbf{A}|^{k-1}(|\bar{\mathbf{A}}||\mathbf{x}_{\tau_z} + \mathbf{v}_{\tau_z} - \hat{\mathbf{x}}_{\tau_z}| + |\mathbf{N}||\mathbf{x}_{\tau_z} + \mathbf{v}_{\tau_z}|) + \mathbf{D}^k|\mathbf{w}_{\tau_z,k}| \\
&\leq \mathbf{D}^k \frac{1}{2}\Delta_{\tau_z} + \mathbf{D}^k|\mathbf{w}_{\tau_z,k}|. \qquad (5.8)
\end{aligned}
$$

To see the last line we let $\mathbf{1}$ denote the column vector with all entries one of appropriate dimension and note that

$$
|\bar{\mathbf{A}}||\mathbf{x}_{\tau_z} + \mathbf{v}_{\tau_z} - \hat{\mathbf{x}}_{\tau_z}| + |\mathbf{N}||\mathbf{x}_{\tau_z} + \mathbf{v}_{\tau_z}| \leq |\bar{\mathbf{A}}|\frac{1}{2}\Delta_{\tau_z} + |\mathbf{N}|\frac{1}{2}\mathbf{K}\Delta_{\tau_z}
$$

Defining $\mathbf{z}_{\tau_z} = |\bar{\mathbf{A}}||\mathbf{x}_{\tau_z} + \mathbf{v}_{\tau_z} - \hat{\mathbf{x}}_{\tau_z}| + |\mathbf{N}||\mathbf{x}_{\tau_z} + \mathbf{v}_{\tau_z}|$ we then see that

$$
\begin{aligned}
\mathbf{z}_{\tau_z}^{j,i} &= \mathbf{A}_{j,i}\frac{1}{2}\Delta_{\tau_z}^{j,i} + \sum_{h=1}^{j-1}|\mathbf{M}_{j,i,h}|\frac{1}{2}\mathbf{K}_h\Delta_{\tau_z}^{h} \\
&\leq \left(|\lambda_{j,i}| + \frac{\delta}{j}\right)\frac{1}{2}\Delta_{\tau_z}^{j,i} + \sum_{h=1}^{j-1}|\mathbf{M}_{j,i,h}|\frac{1}{2}\mathbf{K}_h\frac{\delta}{j n_h \kappa_{j,i,h}}\mathbf{K}_h^{-1}\Delta_{\tau_z}^{j,i,d_{j,i}}\mathbf{1} \\
&\leq \left(|\lambda_{j,i}| + \frac{\delta}{j}\right)\frac{1}{2}\Delta_{\tau_z}^{j,i} + \sum_{h=1}^{j-1} n_h\kappa_{j,i,h}\frac{1}{2}\frac{\delta}{j n_h \kappa_{j,i,h}}\Delta_{\tau_z}^{j,i,d_{j,i}}\mathbf{1} \\
&\leq \left(|\lambda_{j,i}| + \frac{\delta}{j}\right)\frac{1}{2}\Delta_{\tau_z}^{j,i} + \frac{1}{2}\frac{\delta(j-1)}{j}\Delta_{\tau_z}^{j,i} \\
&= \left(|\lambda_{j,i}| + \delta\right)\frac{1}{2}\Delta_{\tau_z}^{j,i}, \qquad (5.9)
\end{aligned}
$$

where we have made use of the bin ordering (5.7). Since (5.9) holds for all $1 \leq j \leq M$ and all $1 \leq i \leq m_j$, it follows that

$$
\mathbf{z}_{\tau_z} \leq \mathbf{D}\frac{1}{2}\Delta_{\tau_z}.
$$

Equation (5.8) now holds from the arguments in the proof of Theorem 4.3.3. Note that the bin ordering (5.7) has an extra $K_{k,\ell}$ term and thus implies the ordering in the multi-sensor case (4.10).

The remainder of the proof now follows directly from the proofs for Theorems 4.3.3 and 4.3.4 from the multi-sensor case. □

Chapter 6

Conclusion

In this report, we have presented a coding and control policy which achieves the minimum rate asymptotically in the limit of large sampling periods for single-station systems driven by Gaussian noise. We have extended our results to a more general class of noise distributions with sufficiently light tails. We have further extended our results to the multi-sensor (single-controller) case under the assumption that each eigenspace is observed by some sensor and to the multi-controller (single-sensor) case under the assumption that each eigenspace is controlled by some controller. In the absence of these assumptions, we have given sufficient conditions for achieving stability.

In all cases, we have established the existence of a unique invariant distribution for the sampled state and a finite second moment of the state. These strong forms of stability have not been considered in the literature for such systems to our knowledge. The proofs use random-time drift criteria for Markov chains.

It is hoped that the results obtained in this work will find applications in a variety of networked control problems.

Part of this report has been submitted to the IEEE Transactions on Automatic Control and part will appear at the IEEE Conference on Decision and Control 2012 in Hawaii.

Chapter 7

Future Work

7.1 Possible Extensions

We have shown that the state can be made positive Harris recurrent with a unique invariant distribution. By the ergodic theorem, we obtain that the transition probability distribution converges to the unique invariant distribution. We have not considered how quickly this convergence takes place. One direction for future work is to study the rate of convergence under the total variation norm and possibly to show that convergence is geometric.

Another direction for further study is to formulate an infinite horizon optimal cost problem. By the ergodic theorem, our stability results show that such cost problems can be well-defined.

The methods in this report are applicable to a variety of decentralized control systems. We provide practical results for systems which have fixed-rate constraints on the communication channels. Noisy channels are significantly more difficult to analyze due to channel errors. See [22] for a discussion. The extension to multi-sensor networks with noisy communication channels (for example, binary erasure channels) linking the sensors to the controller is an open problem.

7.2 Multi-Station Systems

In this section, we give a problem statement for multi-station systems. In the general case, the controllers must send information to each other through signaling (communication via the plant). In the noiseless case, this problem is addressed in [1]. In the noisy case, signaling introduces new complications.

Consider the system

$$\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \sum_{j=1}^{M} \mathbf{B}^j \mathbf{u}_t^j + \mathbf{w}_t, \quad \mathbf{y}_t^j = \mathbf{C}^j \mathbf{x}_t + \mathbf{v}_t^j, \quad 1 \leq j \leq M, \tag{7.1}$$

where $\mathbf{x}_t \in \mathbb{R}^n$ is the state at time $t \in \mathbb{N}$. The control action exerted by controller j at time t is denoted by $\mathbf{u}_t^j \in \mathbb{R}^{m_j}$. The observation made by sensor j at time t is denoted by $\mathbf{y}_t^j \in \mathbb{R}^{p_j}$. The matrices $\mathbf{A}$, $\mathbf{B}^j, \mathbf{C}^j$ and random vectors $\mathbf{w}_t, \mathbf{v}_t^j$ are of compatible size.

We require that $\{\mathbf{w}_t\}$ and each $\{\mathbf{v}_t^j\}$ be sequences of i.i.d. random vectors drawn from a distribution $\mathcal{Z}$, with finite $2+\epsilon$ moments in each component for some $\epsilon > 0$, which admits a probability density that is positive on every open set. At time t, $\mathbf{w}_t$ and each $\mathbf{v}_t^j$ are independent of each other and the state $\mathbf{x}_t$. The initial state, $\mathbf{x}_0$, is drawn from the $\mathcal{Z}$ distribution.

Assumption 7.2.1. *We require joint controllability and joint observability. That is, the pair* $(\mathbf{A}, [\mathbf{B}^1 \quad \mathbf{B}^2 \quad \cdots \quad \mathbf{B}^M])$ *is controllable and the pair* $([(\mathbf{C}^1)^T \quad \cdots \quad (\mathbf{C}^M)^T]^T, \mathbf{A})$ *is observable but the individual pairs* $(\mathbf{A}, \mathbf{B}^j)$ *and* $(\mathbf{C}^j, \mathbf{A})$ *may not be controllable and observable respectively.*

The setup is depicted in Figure 7.1. The observations are made by a set of M sensors and each sensor j sends information to controller j through a finite capacity channel. At each time stage t, we allow sensor $j \in \{1, \ldots, M\}$ to send an encoded value $q_t^j \in \{1, 2, \ldots, N_t^j\}$ to controller j for some $N_t^j \in \mathbb{N}$. The feedback value $b_t \in \{0, 1\}$ is now sent by the plant to all sensors and controllers at times $t = Ts$, where T is the period of our coding policy and $s \in \mathbb{N}$. Note that the plant is a decision maker in this case and must only allow $\mathbf{u}_t^j$ to be applied to the system when all controllers send a non-zero desired control action. That is, $\mathbf{u}_t^j \neq 0$ for all $1 \leq j \leq M$. We define the rate at time t as $R_t = \sum_{j=1}^{M} \log_2(N_t^j)$. The coding scheme is applied periodically with period T and so the rate for all time stages is specified by $\{N_0^j, \ldots, N_{T-1}^j : 1 \leq j \leq M\}$. The average rate is

$$R_{\text{avg}} = \frac{1}{T}\left(2M + \sum_{t=0}^{T-1} R_t\right), \tag{7.2}$$

accounting for the encoded and feedback values.

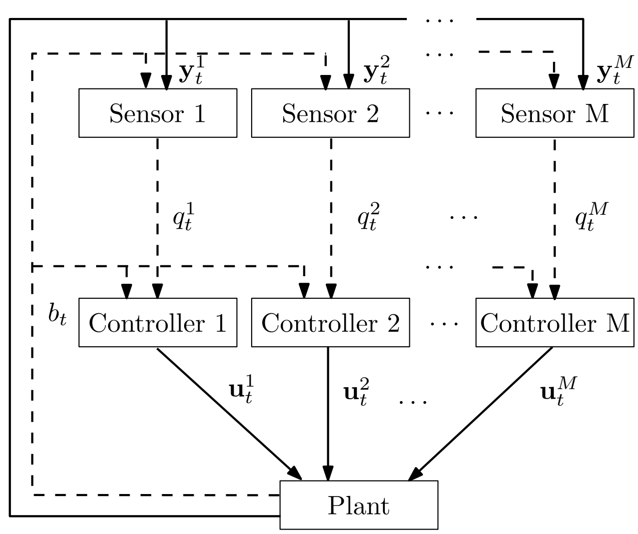

Figure 7.1: A multi-station system with finite-rate communication channels.

Information structure. For a process $\{\mathbf{x}_t\}$ we define $\mathbf{x}_{[a,b]} = \{\mathbf{x}_a, \mathbf{x}_{a+1}, \ldots, \mathbf{x}_b\}$. At time t, each sensor j maps its information $I_t^{s_j} := \{\mathbf{y}_{[0,t]}^j, b_{[0,t-1]}\} \to q_t^j \in \{1, \ldots, N_t^j\}$. Each controller j maps its information $I_t^{c_j} := \{q_{[0,t]}^j, b_{[0,t-1]}\} \to \mathbf{u}_t^j \in \mathbb{R}^{m_j}$.

Consider, for example, a system with two stations. Each station consists of a sensor and a controller, let us call them sensors $1, 2$ and controllers $1, 2$. Sensor 1 receives some information about a component of the state that is controlled by controller 2. Thus, it must send an estimate of the component through the discrete noiseless channel to controller 1, which signals to sensor 2. Let us call the message m. Since the observations of sensor 2 are noisy, it can only recover $m' = m + v$ where v is some noise term. We are interested in the case where the noise is unbounded and there is no guarantee that m' is within the granular region of the quantizer being employed by sensor 2 and controller 2. Thus, we cannot ensure that controller 2 will be able to decode the message correctly. Thus, station 2 receives noisy information and station 1 needs to engage in an information transmission problem over an additive channel with quantized transmissions. This requires a careful analysis.

This simple example shows that the control policy we present in Section 2.3 is insufficient to deal with the multi-station case. At the very least, it would be necessary to introduce a new set of stopping times, to ensure that all messages can be relayed to all controllers. This complicates the analysis considerably and we leave the problem for future work.

The setup presented above leads to the study of a joint source-channel coding problem over an unknown additive channel and the optimality analysis for such problems is difficult. We refer the reader to [22] for a more thorough discussion of some of the issues.

It is clear that for such a class of problems, signaling is essential. The interested reader can find a more in-depth study of signaling in [1] and [10].

Appendix A

Matrices

In this section, we give some simple results from the theory of matrices. We use the lemmas given below in our analysis of single-station, multi-sensor and multi-controller systems.

Lemma A.0.2. *Let* $\mathbf{A} \in \mathbb{R}^n$ *be an invertible matrix. If* $\mathbf{A}$ *has eigenvalues* $\lambda_1, \ldots, \lambda_n$ *then* $\mathbf{A}^{-1}$ *has eigenvalues* $\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n}$*.*

Proof of Lemma A.0.2. If λ_i is an eigenvalue of $\mathbf{A}$ then for some $\mathbf{v} \in \mathbb{R}^n$, $\mathbf{v} \neq 0$ it satisfies

$$(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{v} = 0.$$

Note that

$$\left(-\frac{1}{\lambda_i}\mathbf{A}^{-1}\right)(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{v} = 0$$

$$\left(\mathbf{A}^{-1} - \frac{1}{\lambda_i}\mathbf{I}\right)\mathbf{v} = 0.$$

Thus, $\frac{1}{\lambda_i}$ is an eigenvalue of $\mathbf{A}^{-1}$ and this completes the proof. □

Recall that a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is said to be *symmetric* if $\mathbf{A}^T = \mathbf{A}$. Symmetric matrices have many useful properties. For example, all eigenvalues of a symmetric matrix are real. Also, if $\mathbf{A}^{-1}$ exists, it is symmetric if and only if $\mathbf{A}$ is symmetric. The spectral theorem states that any real symmetric matrix is diagonalizable by an orthogonal matrix. More precisely, there exists some orthogonal matrix $\mathbf{P} \in \mathbb{R}^{n \times n}$

such that

$$\mathbf{P}^T\mathbf{A}\mathbf{P} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$$

where $\mathbf{D}$ is a diagonal matrix. Another way to state the spectral theorem is that the eigenvectors of a symmetric matrix form an orthogonal basis for $\mathbb{R}^n$.

The next lemma follows from the fact that every symmetric matrix has a basis of orthogonal eigenvectors.

Lemma A.0.3. *Let* $\mathbf{A} \in \mathbb{R}^{n\times n}$ *be a symmetric matrix with eigenvalues* $\lambda_1, \ldots, \lambda_n$. *If we let* $\lambda_{\min} = \min\{\lambda_1, \ldots, \lambda_n\}$ *and* $\lambda_{\max} = \max\{\lambda_1, \ldots, \lambda_n\}$ *then* $\lambda_{\min}\mathbf{x}^T\mathbf{x} \leq \mathbf{x}^T\mathbf{A}\mathbf{x} \leq \lambda_{\max}\mathbf{x}^T\mathbf{x}$ *for all* $\mathbf{x} \in \mathbb{R}^n$.

Proof of Lemma A.0.3. From the spectral theorem, there exist a set of orthonormal eigenvectors of $\mathbf{A}$ which span $\mathbb{R}^n$. We label these

$$\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$$

where $\mathbf{e}_i$ is the eigenvector corresponding to λ_i. Let $\mathbf{x} \in \mathbb{R}^n$. We can write

$$\mathbf{x} = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n$$

for some $a_i \in \mathbb{R}$. It then follows that

$$\begin{aligned}
\mathbf{x}^T\mathbf{A}\mathbf{x} &= (a_1\mathbf{e}_1^T + \cdots + a_n\mathbf{e}_n^T)\mathbf{A}(a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n) \\
&= (a_1\mathbf{e}_1^T + \cdots + a_n\mathbf{e}_n^T)(\lambda_1 a_1\mathbf{e}_1 + \cdots + \lambda_n a_n\mathbf{e}_n) \\
&= \lambda_1 a_1^2 + \cdots + \lambda_n a_n^2 \\
&\leq \lambda_{\max}(a_1^2 + \cdots + a_n^2) \\
&= \lambda_{\max}\mathbf{x}^T\mathbf{x}.
\end{aligned}$$

This gives one inequality, the other is clear from the above. □

A symmetric matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is said to be *positive semidefinite* if $\mathbf{x}^T\mathbf{A}\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$. If the inequality is strict for all $\mathbf{x}$ then we say $\mathbf{A}$ is *positive definite*. A matrix is positive semidefinite (positive definite) if and only if all of its eigenvalues are non-negative (positive). Every positive definite matrix is invertible and its inverse is also positive definite.

Appendix B

Stochastic Stability and Markov Chains

We give some basic definitions related to Markov chains and stochastic stability. For a list of definitions on Markov chains, the reader is referred to [11] and [26]. Let $\phi = \{\phi_t, t \geq 0\}$ be a Markov chain defined on a complete separable metric state space $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ with probability space $(\Omega, \mathcal{F}, \mathcal{P})$ where $\mathcal{B}(\mathbb{X})$ denotes the Borel σ-field on $\mathbb{X}$, Ω is the sample space, $\mathcal{F}$ a sigma field of subsets of Ω and $\mathcal{P}$ a probability measure. Let $P(\mathbf{x}, D) := P(\phi_{t+1} \in D | \phi_t = \mathbf{x})$ denote the transition probability from $\mathbf{x}$ to the set D.

Definition B.0.4. *For a Markov chain, a probability measure π is invariant on the Borel space $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ if $\pi(D) = \int_{\mathbb{X}} P(\mathbf{x}, D)\pi(dx)$, for all $D \in \mathcal{B}(\mathbb{X})$.*

Definition B.0.5. *A Markov chain is μ-irreducible if for any set $D \in \mathcal{B}(\mathbb{X})$ with $\mu(D) > 0$ and for all $\mathbf{x} \in \mathbb{X}$ there exists some integer $n > 0$, possibly depending on D and $\mathbf{x}$, such that $P^n(\mathbf{x}, D) > 0$, where $P^n(\mathbf{x}, D)$ is the transition probability in n stages. That is $P(\phi_{t+n} \in D | \phi_t = \mathbf{x})$.*

For any set $D \in \mathcal{B}(\mathbb{X})$, let us define

$$\tau_D = \min\{t \geq 1 : \phi_t \in D\}.$$

Definition B.0.6. *Let μ denote a σ-finite measure on $\mathcal{B}(\mathbb{X})$. A μ-irreducible Markov chain $\{\phi_t\}$ is Harris recurrent if $P(\tau_D < \infty \mid \phi_0 = \mathbf{x}) = 1$ for all $\mathbf{x} \in \mathbb{X}$ and any*

$D \in \mathcal{B}(\mathbb{X})$ satisfying $\mu(D) > 0$. It is positive Harris recurrent if in addition there is an invariant probability measure π.

Definition B.0.7. *A set $S \subset \mathbb{X}$ is small if there is an integer $n \geq 1$ and a positive measure μ satisfying $\mu(\mathbb{X}) > 0$ and $P^n(\mathbf{x}, D) \geq \mu(D)$, for all $\mathbf{x} \in S$ and all $D \in \mathcal{B}(\mathbb{X})$.*

In the following, let $\mathcal{F}_t$ denote the filtration generated by the random sequence $\{\phi_{[0,t]}\}$. Define a sequence of stopping times $\{\mathcal{T}_i : i \in \mathbb{N}_+\}$, measurable on the filtration described above, which is assumed to be non-decreasing, with $\mathcal{T}_0 = 0$.

Theorem B.0.8. *(Theorem 2.1 and Remark 2.1 of [18]) Suppose that we have a μ-irreducible and aperiodic Markov chain ϕ. Suppose moreover that there are functions $V\colon \mathbb{X} \to [a, \infty)$, $\beta\colon \mathbb{X} \to [a, \infty)$, $f\colon \mathbb{X} \to [a, \infty)$, for some $a \geq 0$, a small set C, a constant $b \in \mathbb{R}$ and consider:*

$$E[V(\phi_{\mathcal{T}_{z+1}}) \mid \mathcal{F}_{\mathcal{T}_z}] \leq V(\phi_{\mathcal{T}_z}) - \beta(\phi_{\mathcal{T}_z}) + b1_{\{\phi_{\mathcal{T}_z} \in C\}}, \tag{B.1}$$

$$E\Big[\sum_{k=\mathcal{T}_z}^{\mathcal{T}_{z+1}-1} f(\phi_k) \mid \mathcal{F}_{\mathcal{T}_z}\Big] \leq \beta(\phi_{\mathcal{T}_z}), \quad z \geq 0. \tag{B.2}$$

If $a = 1$ and (B.1) *holds then ϕ is positive Harris recurrent with some unique invariant distribution π. If $a = 0$,* (B.1), (B.2) *hold and ϕ is positive Harris recurrent with some unique invariant distribution π then we get that $\lim_{t\to\infty} E[f(\phi_t)] < \infty$.*

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