Achievable Rates for Stability of LTI Systems over Noisy Forward and Feedback Channels

Serdar Yüksel and Tamer Başar[1]
Coordinated Science Laboratory
University of Illinois at
Urbana-Champaign
1308 West Main S., Urbana, IL

Abstract —

This paper studies sufficient conditions on information rate and control for stability of remote LTI systems connected over noisy forward and reverse (feedback) channels. In particular we consider discrete memoryless (DMC), Gaussian and erasure channels. We study zero-delay, time-invariant coding and decoding policies with memoryless control. For Gaussian channels we provide an achievable region with the optimal linear memoryless policies, and for erasure channels we obtain sufficient packet loss probabilities for stability. For DMCs, we deduce conditions for source and channel codes and controllers leading to an invariant probability density for the state with a finite second moment. We provide a coding scheme achieving stability. All these results reduce to known ones if stochasticities are relaxed. Thus, the paper provides a unified approach to quantization, information rate, sampling and stability for linear networked control systems.

I. Introduction

Research on the interaction of information and control has been rather fruitful, impacting both information and control communities. One problem in this context is the problem of control over communication channels, for which various models for systems and channels have been studied in the recent literature; see e.g. [5], [7], [9], [10] and the references therein. Among these, several studies focused on noiseless systems with time-invariant coders for which the main issue becomes an invariant quantization; see [10], and [23]. Specifically, [10] adopts a Lyapunov approach to stabilize a system and shows that the coarsest quantizer achieving stability is logarithmic and that the design is universal, i.e., it has the same base for construction regardless of the sampling interval. We will show that this property carries over to the stochastic systems as well.

For noiseless channels, structures which are not strictly time-invariant are more rate-efficient ([11], [6], [1], [3], [15], [2]) because the transmitter and receiver can make the updates in the encoding and decoding relying on the data received and knowing that the data sent will make it to the other party with no ambiguity. [16] studies the relationship between information and disturbance rejection. [24] studies the feedback strategies in communication and control building on a Bode-integral argument. For systems with noiseless feedback, the concept of *any-time decoding* has been introduced in [8], where the receiver can halt the process of decoding the sent signal at any appropriate time. Unlike the any-time decoding concept of [8], our encoder and decoders here are not only causal but also of *zero-delay* type [19], i.e., the encoder and decoders encode and decode messages symbol by symbol, and the decoder can only use the past and current received data to generate the estimate. Our motivation for zero-delay coding comes from the fact that in real-time encoding and control the receiver needs that data immediately; for instance in real-time voice over the Internet, UDP is more appropriate than the TCP, for the same reason. Most of the studies mentioned above have considered at least one noiseless channel connecting the controller and the plant and have not touched on the effects when both channels are unreliable. On noisy feedback channels there have been few studies: [14] studies the Gaussian channel case, with no encoding in the reverse channel in the relaxation of the noiseless feedback, [21] studies optimal control policies with packet losses in the feedback channel as well as the forward one; and [22] studies communication with a noisy feedback channel but in the context of estimation, not of control.

In this paper we consider systems where both channels are noisy discrete and memoryless, and with memoryless, time-invariant policies. We focus on stabilizability, for this is a necessary condition for the more comprehensive problem of controllability of linear systems. Here we provide necessary and sufficient rate and coding conditions on the forward and the reverse channels needed to be able to asymptotically stabilize a linear time invariant system. The conditions we provide are tight in the sense that one cannot do any better by relaxing the condition on any one of the channels; thus we generalize in part the results in [6], [10], [7].

The paper is organized as follows. We first start with the system model description and the precise problem formulation in section II. We study discrete memoryless channels, the sufficient rate and coding conditions, and a code construction in section III. Section IV studies Gaussian channels and erasure channels. The paper ends with the concluding remarks of section V.

II. Problem Formulation

We consider in this paper sampled version of an LTI continuous-time system with the scalar dynamics

$$dx_t = (\xi x_t + b' u'_t)dt + dB_t, \qquad (1)$$

where B_t is the standard Brownian motion process, u'_t is the (applied) control which is assumed to be piecewise constant over intervals of length T_s, the initial state x_0 is a second-order random variable; $\xi > 0$, thus the system is unstable without control. After sampling, with period T_s, we have the discrete-time system

$$x_{t+1} = a x_t + b u'_t + d_t \qquad (2)$$

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where x_t is the state at time t, $\{d_t\}$ is a zero-mean i.i.d. Gaussian process. Here $a = e^{\epsilon T_s}$, $b = b'(e^{\xi T_s} - 1)/\xi$ and $E[d_t^2] = \frac{e^{2\xi T_s - 1}}{2\xi}$.

We study the rate conditions for the stability over noisy forward and feedback channels. We refer to the channel carrying signal from the plant to the controller as the *forward channel* and the channel carrying signal from the controller to the plant as the *reverse channel* (see Fig. 1).

In our setup both the plant and the controller act as both transmitters and receivers because of the closed-loop structure. We identify the forward source-channel encoder as a mapping $p_s(z_t|x_t), x_t \in \mathcal{R}$, $z_t \in \mathcal{Z}$, between the source output and channel input. The forward channel is a memoryless stochastic mapping between the channel input and output, $p_c(y_t|z_t), y_t \in \mathcal{Y}$, and the decoder is a mapping between the channel output, the information available at the control, I_{t-1}, and the output, i.e., $p_d(x'_t|I_{t-1}, y_t), x'_t \in \mathcal{X}'$, and $I_t = \{I_{t-1}, y_t, u_{t-1}\}$. The control, $u_t \in \mathcal{U}$, is generated using I_t. The reverse channel also has a source-channel encoder, $p'_s(z'_t|u_t), z'_t \in \mathcal{Z}'$, channel mapping $p'_c(y'_t|z'_t), y'_t \in \mathcal{Y}'$,and a channel decoder $p'_d(u'_t|y'_t), u'_t \in \mathcal{U}'$ (see Fig. 1). For the DMC case, the source-coder is quantizer and the channel encoder generates the bit stream for each of the corresponding quantization symbol, thus generating the joint-source channel encoder.

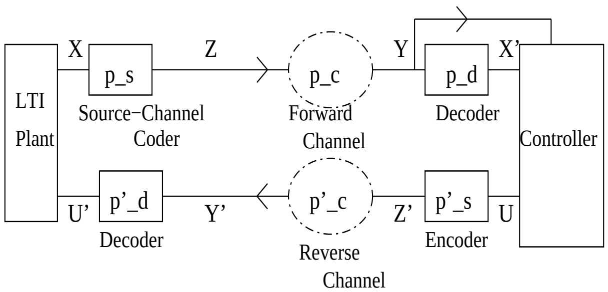

Figure 1: Control over a discrete noisy channel.

For DMC's we use the coding rate definition [13]

$$R_f = \frac{\log(|M_f|)}{N_f},$$

where M_f is the set of plant symbols and N_f is the codelength and likewise for the reverse channel with the subscript changed to r. For a Gaussian system we have the capacity as,

$$C_f = \frac{1}{2}\log(1 + \frac{P_f}{\sigma^2_{w_f}}),$$

where $\sigma^2_{w_f}$ is the noise variance; and likewise for the reverse channel.

Problem: Here we seek to obtain the set of achievable forward and reverse rate pairs leading to a finite state variance:

$$\{R_f, R_r : \lim_{T\to\infty} E[x_T^2] < \infty\}.$$

We say the controller has memory of order m if the information available to it at time t is

$$I_t^m = \{y_{t-m}, \ldots, y_t; u_{t-\max(m,1)}, \ldots, u_{t-1}\}.$$

In case $m = 0$, we will have a memoryless controller; i.e., $I_t^0 = y_t$, which we will study in detail. In this case we will lump the forward source-channel encoder, the forward channel and the decoder mappings as a single mapping $p(x'|x)$ and likewise the reverse source-channel encoder, reverse channel and decoder mappings as $p'(u'|u)$.

A quantizer Q is constructed by corresponding bins $\{\mathcal{B}_i\}$ and their reconstruction levels q_i such that $\forall i$, $Q(x) = q_i <=> x \in \mathcal{B}_i$. We have, $\forall i, q_i \in \mathcal{B}_i$. For scalar quantization, $x \in R$ and $\mathcal{B}_i = (\delta_i, \delta_{i+1}]$, here $\{\delta_i\}$ are termed as "bin edges" and w.l.o.g. we assume the monotonicity on bin edges: $\forall i, \delta_i < \delta_{i+1}$. In this paper we consider "symmetric quantizers", which are defined as: If $\exists$ a quantization bin $(\delta_i, \delta_{i+1}]$, where $0 < \delta_i < \delta_{i+1}$, then $\mathcal{B}_{-i} = [-\delta_{i+1}, -\delta)$ is also a quantization bin.

We define the encodable set $S \in \mathcal{R}$, as the set of elements which are represented by some codeword; $S := \bigcup_i \mathcal{B}_i$. Suppose the state is within the encodable set and is in the ith bin of the quantizer. The source coding output at the plant sensor will represent this state as q_i and send the ith index over the channel. After a joint mapping of the channel and the channel decoder, the controller will receive the index i as index j with probability $p(j|i)$. The controller will apply its control over index j, computing Q'_j -thus the controller decoder, controller and encoder can be regarded as a single mapping- and send it through the reverse channel, which would interpret this value as Q'_l with probability $p'(l|j)$, by a mapping through the reverse channel. Given that the state is in the ith bin, the plant will receive the control Q'_l with probability $\sum_j p'(l|j)p(j|i)$. Thus, the applied control will be $u'_t = Q'_l$ with probability $\sum_j p'(l|j)p(j|i)$, the probability of the state being in the ith bin is $p(i) = p(x \in B_i)$.

While studying the stability of a Markovian system, an appropriate method is to use drift conditions [20]; we will use these conditions to first characterize and then construct our state-encoders. For Gaussian and erasure systems (with an uncountable code alphabet), however, since one does not need to design a quantizer, simpler approaches will be taken.

III. Stabilizing regions for a DMC

We start by providing a necessity condition.

III.A A Necessity Condition

The following theorem [12] provides a necessity condition on the rate requirements.

Theorem III.1 *For the existence of an invariant density with finite variance, channels should satify*

$$\min(C_f, C_r) \geq \log_2(|a|),$$

where C_f and C_r are the forward and the reverse channel capacities.

We now study the conditions for the existence of a finite second moment of the state for systems connected over DMCs.

III.B Stability through Drift Conditions

Consider symmetric quantizers studied before. Suppose a time invariant decoding policy is used by the controller. Then, we have [12].

Theorem III.2 *Let $S \subset X$ be a closed and bounded interval around the origin, $L < \infty$, and let $\delta_i > 0$, $\forall i$ (positive portion of the symmetric quantizer). Finally, let $1_{x \in S}$ be the* *indicator function for x being in S. Then, for a discrete channel, if the following condition holds for all bins*

$$\max(\sum_l \sum_j p(j|i)p'(l|j)[a\delta_i + bQ'_l]^2,$$
$$\sum_l \sum_j p(j|i)p'(l|j)[a\delta_{i+1} + bQ'_l]^2) - \delta_i^2$$
$$< -\epsilon\delta_{i+1}^2 + L1_{\in S} \quad (3)$$

then $\lim_{t\to\infty} E[x_t^2]$ *exists and is finite. The limit distribution is independent of the initial distribution.*

In case the channels are noiseless we obtain a logarithmic quantizer which was, in a control context, first introduced in [10], as we capture with $\epsilon = 0, L = 0$.

Proposition III.1 *Let the forward and the reverse channels be noiseless. Consider a symmetric quantizer. For a scalar system state to satisfy a drift towards the origin, for the non-negative quantizer values quantizer bin edges have to satisfy*

$$\delta_{i+1} \leq (1 + 2/|a|)\delta_i \quad (4)$$

We note that for an arbitrary channel, there does not always exist an invariant density; for instance, for the degenerate case where the channel output is a fixed bin with probability one, there does not exist a drift achieving control and quantization vector. An important observation in the development of this paper is the following [12]:

Proposition III.2 *For a linear system with* $|a| > 1$ *with channel transitions forming an irreducible Markov chain, if the encodable set is bounded, the chain is transient.*

This is an important result for it shows that for the noisy discrete channel case one needs to encode the entire state space and thus, can not use a time invariant fixed length encoding to obtain a finite expected code length, since in that case the codelength would just be the logarithm of the number of symbols, which is unbounded.

It is well known that in order to avoid excessive delays arbitrarily long block codes cannot be used in control systems. Although long block codes improve the channel transmission quality, an increment in the sampling period makes the unstable system more difficult to control. For an LTI system, the effective system matrix will grow exponentially with the sampling period and the uncertainty in the system, measured by the entropy, will grow linearly [2]. The average probability of error decreases exponentially with the length of the code, and the explicit dependence on the length is characterized by the *error exponents* [13]. In this section we will particularly be interested in pairwise errors, that is the probability of error between two different codewords (i.e., $p(m|m'), \quad m \neq m'$). Bhattacharyya distance ([13], Chapter 12), a coding theoretic characterization for the pairwise errors, is lower bounded by the Gilbert bound, i.e. for any two codewords m, m',

$$d(m, m') \geq N[E_L(R) - \epsilon], \quad m \neq m',$$

where R is the coding rate. What this means is that the probability of error between any two (different) codewords ($p(m|m')$) will be upper bounded by $e^{-NE_L^f(R_f)+o(N)}$, where $o(n)/n$ converges to zero as N grows unbounded. Here $E_L(.)$ is the Gilbert lower bound on the error exponent, which gives an upper bound on the probability of errors. Let us fix the forward and reverse channel rates, $R_f = \log_2(M_f)/N_f$ and $R_r = \log_2(M_r)/N_r$. Thus the error exponent function will not change as N_f and N_r increase, and furthermore the increase in N_f, N_r would lead to a higher number of levels in the quantizer and higher number of controller symbols. We also penalize the codelengths in the forward and reverse channels by a linear term in the sampling period; it thus takes longer to send more bits; reliability competes with delay.

Note that the control will only be a function of what it will estimate, it will be at most a one-to-one mapping, thus the number of symbols from the controller to the plant need not be more than the number of symbols it receives from the sensors. We first consider the case where the system (2) is noiseless. We later include noise for a more general setup.

III.C Asymptotic Stability

The following theorem shows that if the controller waits long enough, stability can be achieved.

Theorem III.3 *Suppose a scalar continuous-time system* $\dot{x}_t = \xi x_t + b' u'_t$, *with a bounded initial state* x_0, *is remotely controlled. Let the sampling period be a function of block lengths:* $T_s = \alpha N_f + \beta N_r$; α, β *be possibly depending on the codelengths, and the number of symbols in the state and control be* $K = |\mathcal{X}'| = |\mathcal{U}| = |\mathcal{U}'|$. *Let the rates* $R_f = \log_2(K)/N_f$ *and* $R_r = \log_2(K)/N_r$ *be kept constant as* N_f, N_r *grow. If the system and channel parameters satisfy*

$$(2\xi\alpha - E_L^f(R_f))N_f + (2\xi\beta N_r) < 0$$
$$(2\xi\beta - E_L^r(R_f))N_r + (2\xi\alpha N_f) < 0$$
$$K = e^{N_f R_f} = e^{N_r R_r} > e^{\xi(\alpha N_f + \beta N_r)}, \quad (5)$$

then increasing the sampling period (obtaining additional measurements) decreases a bound on the distortion monotonically. This bound is asymptotically tight and converges to zero.

We have the following remarks:

1. If there is no forward or reverse channel noise, the condition is the well-studied quantization based condition of stability: $K \geq |a|$.
2. Not all information rates are possible. In case there is no noise in the reverse channel for instance, we will need
$$E_L^f(\log_2(K)/N_f) > 2\log_2(|a|),$$
which is the same as
$$\log_2(K) < NE_f^{-1}(2\log_2(|a|)),$$
but also we need $\log_2(K) > \log_2(|a|)$. The solution set can be empty. Note that this resembles the discussion in [7] with regard to the any-time capacity constraint: $C_{anytime}(2\log_2(|a|)) > \log_2(|a|)$.
3. Theorem III.3 shows that the error exponents being positive (which is the case when rate is less than the capacity $R < C$) does not directly lead to stability (as was observed by Sahai [8]), and there needs to be a positive lower bound on the exponent. However, if there is no penalty on the arbitrarily long codelengths through the sampling period, say if $\alpha = t_s/2N, \beta = t_s/2M$, for some constant t_s, then the classical information theoretical results are sufficient; it suffices to have a positive error exponent to achieve stability in the system, which would be the case so long as $R_f < C_f$ and $R_r < C_r$, provided N_f, N_r are sufficiently large.

Thus the achievable rates satisfy (Fig. 2 for $\alpha = \beta = 1$):

$$\frac{1}{N_f}\xi(\alpha N_f + \beta N_r) < R_f < (E_L^f)^{-1}([2\xi\alpha] + 2\xi\beta N_r/N_f)$$
$$\frac{1}{N_r}\xi(\alpha N_f + \beta N_r) < R_r < (E_L^r)^{-1}([2\xi\alpha]N_f/N_r + 2\xi\beta)$$

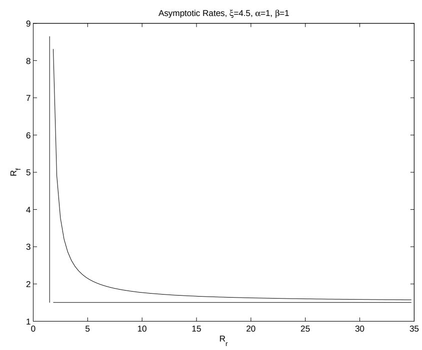

Figure 2: Achievable rates over discrete channels, the area above the curve is achievable. The area below the perpendicular lines is not achievable.

III.D Asymptotic stability with delay restricted codes

We now consider the original system (2) driven by i.i.d. noise, which is a more realistic scheme, where the sampling period is finite, the system is noisy and the amount of data to be sent over a sampling period is finite. In this case the asymptotic analysis becomes inapplicable, and we are forced to obtain a finite design with finite length codes, in a statistical sense.

We know from Proposition III.2 that the encodable set has to be unbounded, and we need to represent this with a finite (in an expectation sense) number of codewords. We will indeed achieve this. In doing so, we will introduce a new coding scheme, where we will transmit the coset of the code and send the particular bin by an additional channel which is *noisy* as well. We quantify the requirements needed by this channel.

Suppose we have $K = 2^{N_f R_f}$ symbols that we can transmit during each time stage. We will partition the entire state space into bins and group K adjacent elements into one larger bin, indexed by I, and represent them by a single channel codebook. We refer to these ensemble of bins by a *Code Bin*. Hence, a total of $2^{N_f R_f}$ codewords are used to represent the entire state space (see Figure 3). Thus we have,

$$CodeBin(I) = \{x : \delta_{I N_f R_f} \leq x < \delta_{(I+1) N_f R_f}\}$$

We denote the bin indices by δ_{nI+i}, which means the edge belongs to code bin I and is represented by the *ith* channel codeword. We say the source code is in mode I, if the state is in Code Bin I (see Figure 4). The reconstruction value of each bin is assumed to be the midpoint, such that $Q_i = 1/2(\delta_i + \delta_{i+1})$. We define $p^m(J|I)$ as the probability of error of CodeBin (mode of the quantizer) transmission from mode I to mode J through the side channel. This means the mode is errorenously transmitted from the plant to the controller if $J \neq I$. Likewise for the feedback channel we have $p^{m'}(L|J)$ as the side channel mapping.

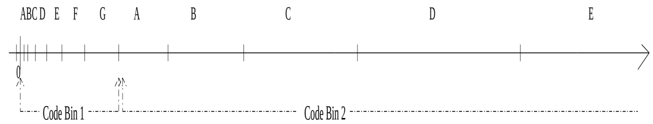

Figure 3: Illustration of the binning approach to the joint source channel code; the symbols in a given Code Bin are represented by the same channel code -letters A, B, C, ... - ; the mode symbol -1,2,3 ... -is carried by the side channel.

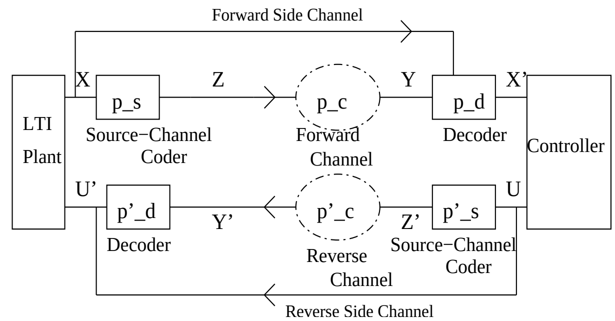

Figure 4: Side channels carry the mode of the source mapping, the system is tolerant to the errors in the side channel as well.

Theorem III.4 *Suppose the scalar continuous-time system $dx_t = (\xi x_t + b'u'_t)dt + dB_t$, is remotely controlled. Let T_s be a sampling period which is function of block lengths: $T_s = \alpha N_f + \beta N_r$, α, β be possibly depending on the codelengths, and the number of symbols in the state and control be $K = |\mathcal{X}'| = |\mathcal{U}| = |\mathcal{U}'|$. Suppose the forward and reverse channel codes are of N_f and N_r bits long, and let the rates be $R_f = log_2(K)/N_f$ and $R_r = log_2(K)/N_r$. Define $T(\gamma, p^m, I)$ as*

$$\sum_{L, L\neq I} \sum_{J} P^{m'}(L|J) P^m(J|I) 4\gamma^2 \gamma^{2\max(0, N_f R_f(|L|-|I|))}$$

$$U(\gamma, I) = \gamma^2 \sum_{L} \sum_{J} P^{m'}(L|J) P^m(J|I) \gamma^{2\max(0, N_f R_f(|L|-|I|+1))}$$

$$Z = [e^{-N_f E_L^f(R_f) - N_r E_L^r(R_r)} 2^{N_f R_f} + e^{-N_f E_L^f(R_f)} + e^{-N_r E_L^r(R_r)}]$$

If for some $\gamma > 1$ the forward, reverse and side channels satisfy the following

$$\overline{\lim}_{I\to\infty} U(\gamma, I) =: \bar{U}(\gamma) < \infty$$
$$\overline{\lim}_{I\to\infty} T(\gamma, p^m, I) =: T(\gamma, p^m) < 1$$
$$\gamma < 1 + 2(e^{-\xi})^{\alpha N_f + \beta N_r} \cdot \sqrt{[(1-\epsilon) - 4Z 2^{N_f R_f} \bar{U}(\gamma) - T(\gamma, p^m)]}, \quad (6)$$

then drift conditions hold and there exists a coding scheme leading to a finite second moment. The source coder will be a symmetric logarithmic quantizer with expansion ratio γ, i.e., $|\delta_{i+1}| < \gamma|\delta_i|$.

III.E On the necessity of the Side Channel

One might not need the side channel for the forward channel. The conceptual idea is based on the Slepian-Wolf coding theorem, which was applied in [4] as uniform binning in a decentralized linear system (see also the recent study [17]). We have studied this in [12], with a discussion on the reverse channel, which requires a different treatment.

IV. Gaussian and Erasure Channels

IV.A Stabilizing Rates over Gaussian Channels

We now study Gaussian channels. For Gaussian channels we associate power constraints with the encoder outputs, P_f and P_r, for the forward and the reverse channels, respectively (Fig. 5).

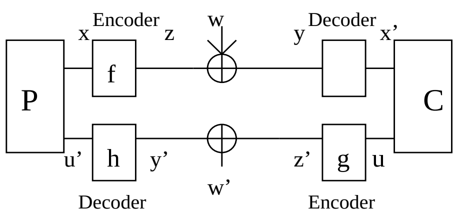

Figure 5: Control over Gaussian channels, the noise terms are w_f and w_r'.

For Gaussian channels, one does not face the difficulties of explicitly using a finite codelength, for $\mathcal{X}'$ is the entire real line, and the improbable events are already sent out with higher magnitude signals, whose contribution to the expected power is limited; this is the main difficulty one faces in the design of variable length channel codes in a control context. The impropable events ought to be represented by longer codelengths to avoid transience. For Gaussian channels, one does not need to use drift analysis either since one can obtain achievable rate regions by adopting linear policies. Below we obtain the optimal linear memoryless encoder and decoders for the minimization of the steady-state variance.

Theorem IV.1 *Let the forward and the reverse channels have capacities C_f and C_r, respectively. The optimal memoryless linear policies for the minimization of the steady state variance are:*

$$z_t = f(x_t) = \sqrt{P_f} x_t / ||x_t||$$
$$z_t' = f'(z_t + w_t^f) = [\sqrt{P_f}/(P_f + {\sigma_w^f}^2)](z_t + w_t^f)$$
$$u_t' = h(y_t') = -a\sqrt{P_f P_r}\left[||x_t||/[\sqrt{P_f + {\sigma_w^f}^2}(P_r + {\sigma_w^r}^2)]\right] y_t'$$

where $y_t' = z_t' + w_t^r$ and $||x_t|| = \sqrt{E[x_t^2]}$. If the forward and the reverse channels satisfy:

$$2^{-2C_f} + 2^{-2C_r} - 2^{-2C_f - 2C_r} < 1/a^2, \tag{7}$$

then the steady state variance is finite.

Proof: We first have

$$E[x_{t+1}^2] = [(a + \rho\eta_f\eta_r)^2 + \rho^2\eta_f^2\eta_r^2\sigma_w^2/P_f + \rho^2\eta_f^2\eta_r^2\sigma_w'^2]E[x_t^2] + E[d_t^2], \tag{8}$$

where ρ is a scaling term and

$$\eta_f = P_f/(P_f + \sigma_w^2),$$

likewise for η_r. The coefficients are then optimized by the selection of the gain terms leading to the smallest steady-state variance. Optimal gain ρ turns out to be $-a$, as to be expected by intuition. Upon recognizing the signal to noise ratio and capacity in η_f, η_r, one obtains the given result. $\diamond$

Remark. [14] studied Gaussian channels in the context of relaxation of the 'equimemory' condition. In [14] control is not encoded and by optimizing over the controller gain it is proven that the rate in the reverse channel does not improve the performance after a threshold value. In our case, however, the control is encoded as well, and an increment in power strictly improves the rate restrictions for the forward channel. Note that if one encodes the control, the problem is that of an information transmission problem. If there is no encoding, then control is applied as it is received and there exists an optimal value for the control power. $\diamond$.

We now plot the achievable (sufficient) and the necessary rate regions (for Gaussian channels) in Fig 6.

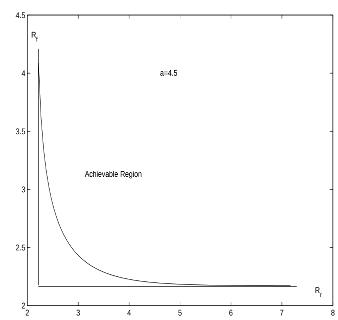

Figure 6: Forward and reverse channels trade-off for Gaussian channels. The area above the convex curve is achievable by linear memoryless policies, the areas outside the perpendicular edges are not achievable by any scheme; in the figure we have picked $a = 4.5$.

IV.B Stabilizing Loss Probabilities

Suppose we have erasure forward and reverse channels which lose packets with probabilities p_f and p_r, respectively. Consider the case where the packets can be sent without a need of quantization, i.e., the erasure channel codebook set is the real line (thus the capacity is indeed infinite). In this case we have the following result.

Proposition IV.1 *Consider the unstable plant in (2). The quantity $\lim_{t\to\infty} E[x_t^2]$ is finite, if the forward and the reverse channel packet loss probabilities satisfy:*

$$p_f + p_r - p_f p_r < 1/a^2. \tag{9}$$

Proof: We again consider memoryless policies. We will have with probability $(1 - p_f)(1 - p_r)$

$$x_{t+1} = ax_t - b(a/b)x_t + d_t.$$

If there is a loss in the forward channel as well as the reverse channel, zero control is applied (as a consequence of the memoryless policy). Then the evolution of the second moment will be

$$E[x_{t+1}^2] = [1 - (1 - p_f)(1 - p_r)]a^2 E[x_t^2] + E[d_t^2] \tag{10}$$

This concludes the proof. ⋄

Remark Note the resemblance of (9) with what was obtained in [21] as a sufficient condition on stabilizability over UDP protocols with full observations in case of a successful transmission. One can compare this result with that of TCP to conclude that there is a loss due to the absence of the noiseless feedback. In case of TCP what we needed there was $\max\{p_f, p_r\}a^2 < 1$. ⋄

V. Conclusion and Future Work

In this paper, we have analyzed control systems where feedback loops are closed over noisy channels. We studied time-invariant encoding policies and memoryless control. For DMCs, we used negative stochastic drift conditions towards a small set around the origin and we provided rate conditions for achieving this objective. We also studied the interplay between information rate, channel error exponents and the sampling rate. We provided rate conditions for Gaussian channels, with optimal linear memoryless policies. The packet loss probability pairs tolerable by the system connected over a communication network was also studied. The unifying result that appears to different channels is that there is an additional price to be paid if there is unreliability in the reverse channel as well. The paper also shows that using memoryless policies is not too bad, for the achievable rate regions are not too far off the unachievable ones. We provided a framework to encode the state-space for dynamic systems.

Here we have used memoryless schemes. One future direction for research would be to quantify the improvement due to use of memory in the controller.

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