ON THE EXISTENCE OF OPTIMAL POLICIES FOR A CLASS OF STATIC AND SEQUENTIAL DYNAMIC TEAMS*

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Abstract. In this paper, we identify sufficient conditions under which static teams and a class of sequential dynamic teams admit team-optimal solutions. We first investigate the existence of optimal solutions in static teams where the observations of the decision makers are conditionally independent given the state and satisfy certain regularity conditions. Building on these findings and the static reduction method of Witsenhausen, we then extend the analysis to sequential dynamic teams. In particular, we show that a large class of dynamic linear-quadratic-Gaussian (LQG) teams, including the vector version of the well-known Witsenhausen's counterexample and the Gaussian relay channel problem viewed as a dynamic team, admit team-optimal solutions. Results in this paper substantially broaden the class of stochastic control problems with nonclassical information known to have optimal solutions.

Key words. team theory, distributed control, decentralized control, asymmetric information, optimal stochastic control

1. Introduction. A team is a group of agents acting collectively, but not necessarily sharing all the information, in order to optimize a common cost function. In stochastic teams, there may be probabilistic uncertainty about initial state, observations, or cost functions, and/or the evolution of the state is disturbed by some external noise process. The statistics of the noise processes, state evolution equations, and observation equations are common knowledge among the agents of a team. At every time step, each agent acquires some information about the state via observation, may acquire other agents' past and current observations and past actions, and may recall past observations and actions. If each agent's information is determined only by primitive/exogenous random variables, the team is said to be a static team. If at least one agent's information is affected by an action of another agent, the team is said to be dynamic. The information structure in a team determines such functional and probabilistic relations in that team.

A system in which there is a prespecified order of actions is said to be a sequential team. In nonsequential teams, the ordering of "who acts when" is not known a priori, and ordering of actions may be determined by the outcome of some random process [30]. In this paper, we focus our attention on the existence of optimal strategies in sequential dynamic teams. A sequential team in which the information available to the decision makers in forward time is noncontracting is said to be a classical team

(such settings include the well-studied single-agent Markov decision problems with full memory). A sequential team, which is not classical but has the property that whenever an agent's, say $\mathbf{A}k$'s, information is affected by the action of some other agent, say $\mathbf{A}j$, $\mathbf{A}k$ has access to $\mathbf{A}j$'s information, is said to have a quasi-classical (or partially nested) information structure. An information structure which is not quasi-classical is said to be nonclassical. For an extensive discussion on classifications of information structures, the reader is referred to [34] and [37].

We next provide a brief overview of the early developments in the area of teams with asymmetric information, as is relevant to this paper. In 1962, Radner published a seminal paper on team decision problems [28], where he showed that a class of static stochastic teams with continuously differentiable and strictly convex cost functions admit globally optimal (minimizing) solutions which are also the unique person-by-person-optimal solutions; this result was later extended in [22] to settings with exponential cost functions. A by-product of Radner's result is that in static linear-quadratic-Gaussian (LQG) team problems, linear strategies of the agents that are person-by-person-optimal are also team-optimal. Furthermore, partially nested LQG teams also admit linear optimal solutions, as was observed in [21, 12]. When the information structure is nonclassical, however, Witsenhausen showed that even seemingly simple LQG settings can be very difficult to solve [33]: He devised a scalar LQG team problem which admits an optimal solution, which, however, is not linear.

Obtaining solutions of classical dynamic team problems is quite well understood, with dynamic programming providing the most convenient approach. For dynamic teams not of the classical type, however, there is no universally applicable systematic approach; see [26, 37, 25] for detailed coverage and analysis of the various solution approaches, which also depend on the underlying information structure.

In this paper, we prove that a class of sequential teams with a certain information structure (not necessarily classical or quasi-classical) admits team-optimal solutions. This constitutes a first step toward understanding the most general conditions under which stochastic dynamic team problems admit optimal solutions. The results obtained and the general framework adopted are applicable to various models of team problems that are studied in economics, information theory, network information theory, and stochastic control.

1.1. Previous work. Some of the earliest and most fundamental works on understanding the role of information in general dynamic stochastic teams were carried out by Witsenhausen in [34, 33, 35, 32]. Among these fundamental contributions, in [32] Witsenhausen showed that all sequential team problems satisfying an absolute continuity condition of certain conditional measures can be transformed into an equivalent static team problem with a different cost function and with the agents observing mutually independent random variables. This transformation of a dynamic team into a static team problem with independent observations is called "static reduction" of the dynamic team problem (see section 3.7 and, particularly, page 114 of [37] for both a discussion on this reduction and an overview of Witsenhausen's contributions). In this paper, we make use of this equivalence between dynamic and static team problems to show the existence of optimal strategies in the original dynamic team problem. Recently, [11] studied continuous-time stochastic team decision problems that are partly driven by Brownian motion, and showed that the static reduction of the dynamic team problem can be carried out using the Girsanov transformation.

The existence of optimal strategies for Witsenhausen's counterexample was proved in [33], where his proof relied on the structure of the cost function of the team. In

that problem, the unique optimal strategy of the second agent involves the conditional mean of the control action of the first agent using an observation that is an additive noise corrupted version of the control action of the first agent, where the noise is a zero-mean unit variance Gaussian random variable. When substituted back, this makes the expected cost functional of the team a nonconvex functional on the space of strategies of the first agent. Witsenhausen used several techniques from real analysis to show that an optimal strategy of the first agent exists when the initial state of the system is an arbitrary second-order random variable. A shorter proof of existence for this problem was later presented in [36] using tools from optimal transport theory. One variant of this problem is the Gaussian test channel; there are other variants as well, all with nonclassical information [4]. For the Gaussian test channel, proof of existence of optimal strategies (and their derivations) is indirect. First, the cost function is lower bounded using the data processing inequality [13], and then explicit linear strategies are constructed which make the cost achieve the lower bound.[1] Proofs of existence of optimal strategies in some other teams with nonclassical information using this method can be found in [5].

The main difference between the formulations of Witsenhausen's counterexample and the Gaussian test channel is in the cost functions (even though they are both quadratic) [5, 4]. In the case of Witsenhausen's counterexample, the explicit forms of optimal strategies are not known, whereas in the case of the Gaussian test channel, linear strategies are optimal; in both problems (where all random variables are jointly Gaussian), optimal solutions exist. If, however, the distributions of the primitive random variables are discrete with finite support, there is no optimal solution even in the class of behavioral strategies of the agents, as illustrated in [37, p. 90]. Thus, the question of sufficient conditions for a team problem to admit an optimal solution is important and is addressed in this paper.

Several authors have proven the existence of optimal solutions to stochastic optimization problems through "lifting." Specifically, the problem of optimizing an expected cost functional on the space of Borel measurable functions is lifted to an equivalent optimization problem in which the cost functional is defined on the space of probability measures. This technique is heavily used in proving the existence of optimal strategies in Markov decision processes; see for example, [27, 20, 6], among several others. This is also the central concept for studying optimal transport problems [2, 31], where the cost function is a measurable function of two random variables with given distributions, and the optimization is performed on the space of joint measures of the random variables given the marginal distributions. In this paper, we use a similar lifting technique to establish the existence of optimal strategies in certain classes of static and dynamic teams.

A variant of the problem of existence of optimal strategies in stochastic dynamic teams is that of the existence of optimal observation channels in such systems. The relevant question there is how to design observation channels (for example, quantizers) in a team problem so that the overall expected cost is minimized. This problem was studied in [38], where some sufficient conditions were obtained on teams and sets of observation channels to ensure the existence of optimal quantizers. There again, the problem of designing quantizers was lifted to one of designing joint probability measures over the state and the observation of an agent satisfying certain constraints.

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[1]Recently, [23] showed that using mutual information to lower bound the expected cost in the Gaussian test channel is akin to convexifying the nonconvex constraint in an optimization problem over joint measures.

In [38], a topology of information channels was introduced based on a fixed input distribution. Related approaches can be found in [10, 9]. In this paper, we further generalize these approaches.

1.2. Outline of the paper. We prove, under some sufficient conditions, the existence of optimal strategies in a class of sequential teams. A general stochastic dynamic team problem is formulated in section 2. Thereafter, we study two related static teams in sections 3 and 4. In section 3, we show that if the cost function of the team is continuous and bounded, and action spaces are compact, then under mild conditions on the observation kernels, a team-optimal solution exists. In section 4, we extend the result to the case when the action spaces may not be compact and the cost function of the team is a nonnegative continuous, possibly unbounded, function. In section 5, we show, using the static reduction technique of Witsenhausen [32], that a large class of dynamic teams admit team-optimal solutions. We use this result to show, in section 6, that most LQG teams with no sharing of observation admit team-optimal solutions. In section 7, we prove the existence of optimal strategies in several LQG teams of broad interest. Finally, we present concluding remarks in section 8.

1.3. Notation. We introduce here some of the notation used throughout the paper. For a natural number N, we let $[N]$ denote the set $\{1, \ldots, N\}$. The set of all nonnegative real numbers is denoted by $\mathbb{R}^+$. If $\mathcal{X}$ is a set and $\mathcal{A}$ is a subset of $\mathcal{X}$, then $\mathcal{A}^{\complement}$ denotes the complement of the set $\mathcal{A}$.

We use boldface letters $\mathbf{a}, \mathbf{b}, \ldots$ to denote generic elements in sets $\mathcal{A}, \mathcal{B}$, and so on. If the space $\mathcal{X}$ is the real space, then we simply use x to denote a generic element of $\mathcal{X}$, with no boldface. Uppercase boldface letters, for example, $\mathbf{X}$, are used to denote random variables. A superscript denotes the index of an agent, while a subscript denotes the time step or an index of a sequence. For example, $\mathbf{U}_t^i$ denotes the control action taken by agent i at time step t. If $\mathbf{a} \in \mathbb{R}^n$, then we use $\|\mathbf{a}\|_R^2$ to denote $\mathbf{a}^{\mathsf{T}} R \mathbf{a}$ for a positive-definite matrix R.

Let X be a nonempty set. For a set of elements $\{x_1, \ldots, x_n\}$ in X, we let $x_{1:n}$ denote this set. The set $\{x_1, \ldots, x_n\} \setminus \{x_i\}$ is denoted by x_{-i}. If $X_1, \ldots, X_n$ are nonempty sets, then $X_{1:n}$ and X_{-i} are shorthand notation for the product sets $X_1 \times \cdots \times X_n$ and $X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_n$, respectively. If we write $x_{1:n} \in X_{1:n}$, then it is construed as $x_1 \in X_1$, $x_2 \in X_2$, and so on. Similarly, $x^{1:n}$ denotes the set $\{x^1, \ldots, x^n\}$, and $x_{1:t}^{1:n}$ denotes $\{x_1^1, \ldots, x_1^n, \ldots, x_t^1, \ldots, x_t^n\}$, where each element is in the appropriate space.

Let $\mathcal{X}$ be a topological space. The vector space of all bounded continuous functions on $\mathcal{X}$ endowed with the supremum norm $\|\cdot\|_\infty$ is denoted by $C_b(\mathcal{X})$, that is, $C_b(\mathcal{X}) := \{f : \mathcal{X} \to \mathbb{R} : f \text{ is continuous and } \|f\|_\infty < \infty\}$. The vector space $C(\mathcal{X})$ is the space of all continuous functions on the topological space $\mathcal{X}$, which are possibly unbounded. Thus, $C_b(\mathcal{X}) \subset C(\mathcal{X})$. If $\mathcal{X}$ is a metric space, then the vector space $U_b(\mathcal{X}) \subset C_b(\mathcal{X})$ denotes the set of all bounded uniformly continuous functions on $\mathcal{X}$. The Borel σ-algebra on $\mathcal{X}$ is denoted by $\mathfrak{B}(\mathcal{X})$. The spaces $\mathsf{ca}(\mathcal{X})$ and $\wp(\mathcal{X})$ denote, respectively, the vector space of countably additive signed measures on $\mathcal{X}$ and the set of all probability measures on $\mathcal{X}$ endowed with weak* topology. If $\mu^i \in \wp(\mathcal{A}^i), i \in [N]$, are probability measures, then $d\mu^{1:N}$ denotes the product measure $\mu^1(d\mathbf{a}^1) \cdots \mu^N(d\mathbf{a}^N)$. We let $\mathbb{1}_{\{\cdot\}}$ denote the Dirac probability measure over the point $\{\cdot\}$. Let $\mathcal{A}$ and $\mathcal{B}$ be Polish spaces and let $\mu \in \wp(\mathcal{A} \times \mathcal{B})$. Then, $\mathsf{Pr}_{\#}^{\mathcal{A}} \mu$ denotes the marginal of μ on space $\mathcal{A}$, that is, for any Borel set $\mathcal{A} \subset \mathcal{A}$, $\mathsf{Pr}_{\#}^{\mathcal{A}} \mu(\mathcal{A}) := \mu(\mathcal{A} \times \mathcal{B})$.

2. Problem formulation. Within a state-space model, we consider an N-agent dynamic team with no observation sharing information structure. Let the state at time instant $t \in [T]$ be denoted by $\mathbf{X}_t$, and let the space of all possible states at time t be denoted by $\mathcal{X}_t$. The action of agent i at time t lies in a space $\mathcal{U}_t^i$, and its action is denoted by $\mathbf{U}_t^i$. The agents observe the plant states and past control actions through noisy sensors (or channels), and the observation of agent i at time t is denoted by $\mathbf{Y}_t^i$, which lies in a space $\mathcal{Y}_t^i$. Throughout the paper, the spaces $\mathcal{X}_t, \mathcal{U}_t^i, \mathcal{Y}_t^i$ are assumed to be complete separable metric spaces (also called Polish spaces, such as $\mathbb{R}^n$ or the space of measures over $\mathbb{R}^n$) at all time $t \in [T]$ and for all agents $i \in [N]$.

The state of the system evolves according to

(2.1) $$\mathbf{X}_{t+1} = \tilde{f}_t(\mathbf{X}_{1:t}, \mathbf{U}_{1:t}^{1:N}, \mathbf{W}_t^0), \quad t \in [T],$$

where $\mathbf{W}_t^0$ is the actuation noise on the system. We denote the space of all possible actuation noises by $\mathcal{W}_t^0$. Agent i at time $t \in [T]$ makes an observation, which depends on past states and control actions, according to

(2.2) $$\mathbf{Y}_t^i = \tilde{h}_t^i(\mathbf{X}_{1:t}, \mathbf{U}_{1:t-1}^{1:N}, \mathbf{W}_t^i),$$

where $\mathbf{W}_t^i$, which takes values in the space $\mathcal{W}_t^i$, is the observation noise of agent i at time t. We again assume that $\mathcal{W}_t^i, i \in \{0\} \cup [N], t \in [T]$, are Polish spaces. We make the following assumption on the state transition functions and the observation functions of the agents.

ASSUMPTION 2.1. *The state transition functions $\tilde{f}_t$ and observation functions $\tilde{h}_t^i, i \in [N]$, are continuous functions of their arguments for all time steps $t \in [T]$.*

The random variables $\{\mathbf{X}_1, \mathbf{W}_{1:T}^{0:N}\}$ are primitive random variables and are assumed to be mutually independent. We let $\xi_{\mathcal{X}_1}$ denote the probability measure on $\mathcal{X}_1$ and $\xi_{\mathcal{W}_t^i}$ denote the probability measure on $\mathcal{W}_t^i$ for $i \in \{0\} \cup [N]$ and $t \in [T]$.

2.1. Information structures and strategies of the agents. At each instant of time, we assume that the only information each agent acquires is its own observation. Each agent uses its information to determine its control action. Toward this end, we allow the agents to act in a predetermined fashion, and when they act they choose either a deterministic strategy or a randomized (more precisely behavioral) strategy. We define these two notions of strategies below.

DEFINITION 2.1 (deterministic strategy). *A deterministic strategy for an agent i at time t is a Borel measurable map $\gamma_t^i : \mathcal{Y}_t^i \to \mathcal{U}_t^i$. Let $\mathcal{D}_t^i$ be the space of all such maps, which we call the deterministic strategy space of agent $i \in [N]$ at time $t \in [T]$.*

DEFINITION 2.2 (behavioral strategy). *A behavioral strategy of agent i at time t is a conditional measure π_t^i satisfying the following two properties:*
  1. *For every $\mathbf{y}_t^i \in \mathcal{Y}_t^i$, $\pi_t^i(\cdot|\mathbf{y}_t^i) \in \wp(\mathcal{U}_t^i)$.*
  2. *For every $\mathcal{U} \in \mathfrak{B}(\mathcal{U}_t^i)$, $\mathbf{y}_t^i \mapsto \pi_t^i(\mathcal{U}|\mathbf{y}_t^i)$ is a $\mathfrak{B}(\mathcal{Y}_t^i)$-measurable function.*
Let $\mathcal{R}_t^i$ denote the behavioral strategy space of agent $i \in [N]$ at time $t \in [T]$.

Remark 2.1. For an agent at any time step, any deterministic strategy is by definition also a behavioral strategy. For example, if γ_t^i is a deterministic strategy of agent i at time t, then the corresponding (induced) behavioral strategy is $\pi_t^i(d\mathbf{u}_t^i|\mathbf{y}_t^i) = \mathbb{1}_{\{\gamma_t^i(\mathbf{y}_t^i)\}}(d\mathbf{u}_t^i)$.

As a consequence of the remark above, throughout this paper we will work with behavioral strategies of an agent with the understanding that this also covers deterministic strategies of that agent as well.

2.2. Expected cost functional of the team. The team is equipped with a cost function $\tilde{c}$, which is assumed to be a nonnegative continuous function of all states $\mathbf{X}_{1:T+1}$, observations $\mathbf{Y}_{1:T}^{1:N}$, and actions $\mathbf{U}_{1:T}^{1:N}$ of the agents. However, we can substitute (2.1) recursively so that the cost function becomes purely a function of the primitive random variables $\{\mathbf{X}_1, \mathbf{W}_{1:T}^0\}$, observations $\mathbf{Y}_{1:T}^{1:N}$, and the control actions of all agents, $\mathbf{U}_{1:T}^{1:N}$. Therefore, for a fixed realization of the primitive random variables, observations and control actions of the agents, the cost incurred by the team can be written as $c(\mathbf{x}_1, \mathbf{w}_{1:T}^0, \mathbf{y}_{1:T}^{1:N}, \mathbf{u}_{1:T}^{1:N})$ for some c, which is clearly related to $\tilde{c}$. We have the following result on the cost function c.

LEMMA 2.3. *If Assumption* 2.1 *holds, then the cost function c is continuous.*

Proof. Note that by construction, c is generated by $\tilde{c}$ as

$$c(\mathbf{x}_1, \mathbf{w}_{1:T}^0, \mathbf{y}_{1:T}^{1:N}, \mathbf{u}_{1:T}^{1:N}) = \tilde{c}(\mathbf{x}_1, \tilde{f}_1(\mathbf{x}_1, \mathbf{u}_1^{1:N}, \mathbf{w}_1^0), \dots, \tilde{f}_T(\mathbf{x}_{1:T}, \mathbf{u}_{1:T}^{1:N}, \mathbf{w}_T^0), \mathbf{y}_{1:T}^{1:N}, \mathbf{u}_{1:T}^{1:N}),$$

where $\mathbf{x}_t$ is substituted as a function of $\mathbf{x}_1, \mathbf{w}_{1:t-1}^0$ and $\mathbf{u}_{1:t-1}^{1:N}$ using (2.1) for all $t \in [T]$. Since $\tilde{c}$ and $\{f_t\}_{t \in [T]}$ are continuous functions of their arguments, we conclude that c is a continuous function on $\mathcal{X}_1 \times \mathcal{W}_{1:T}^0 \times \mathcal{Y}_{1:T}^{1:N} \times \mathcal{U}_{1:T}^{1:N}$. $\qquad\square$

In standard optimal control problems, the cost function of the team is taken to be a sum of stagewise cost functions, in which the cost function at every time step depends on the current state and actions of the agents. However, we do not assume such a structure on the cost function of the team considered in this paper. This general cost function encompasses ones that appear in certain classes of communication systems, economic systems, and feedback control over noisy channels.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the underlying probability space, possibly depending on the choice of randomized strategies of the agents. Throughout this paper, we use $J : \mathcal{R}_{1:T}^{1:N} \to \mathbb{R}_+$ to denote the expected cost functional of the team, which is defined as

$$J(\pi_{1:T}^{1:N}) = \mathbb{E}\left[c(\mathbf{x}_1, \mathbf{w}_{1:T}^0, \mathbf{y}_{1:T}^{1:N}, \mathbf{u}_{1:T}^{1:N})\right],$$

where the expectation is taken with respect to the measure induced on the random variables by the choice of behavioral strategies $\pi_{1:T}^{1:N}$. We make the following natural assumption on the team described above.

ASSUMPTION 2.2. *There exists a set of behavioral strategies $\tilde{\pi}_{1:T}^{1:N} \in \mathcal{R}_{1:T}^{1:N}$ of the agents, which results in finite expected cost to the team.*

2.3. Solution approach and the proof program. Our proof of existence of optimal strategies in the team problem formulated above follows three steps. We first show the existence of optimal strategies in a static team in section 3, in which (i) the cost function of the team is a continuous and bounded function of its arguments, (ii) the action spaces of the agents are compact, and (iii) the observation channels of the agents satisfy a technical assumption. We refer to this static team as team **ST1**. We establish a tightness result on the joint measures over state, observation, and action spaces of each agent. For any sequence of joint measures induced by behavioral strategies of the agents that achieves expected costs converging to the infimum of the expected cost of the team, we show that there exists a convergent subsequence of joint measures, which are induced by a set of behavioral strategies, whose limit achieves the infimum of the expected cost functional of the team. Next, we show the existence of optimal strategies in a static team in which (i) the action spaces of the agents are noncompact and (ii) cost function of the team is continuous and has a coercive structure. We refer to this static team as team **ST2**. This result is established in section 4 using the results of section 3.

Subsequently, we use Witsenhausen's static reduction technique to reduce a sequential dynamic team into a (reduced) static team with independent observations of the agents. We assume a certain structure on the cost function of the dynamic team. A challenge with this approach is that the cost function of the reduced static team problem may not satisfy a coercivity condition (introduced later) even though the cost function of the dynamic team satisfies that coercivity condition. To alleviate this problem, we provide a novel approach by restricting the search for optimal behavioral strategies to a compact set. The dynamic team problem is then solved under some mild assumptions that are delineated in section 5.

In the next section, we establish the existence of team-optimal solutions to the first static team problem mentioned above.

3. Existence of optimal solution in ST1. Here, we study the N-agent static team in which each agent i observes a random variable $\mathbf{Y}^i$, correlated with the random variable $\mathbf{X}$, and takes an action $\mathbf{U}^i$. We let $\mathcal{X}$ denote the state space of the team, and $\mathcal{Y}^i$ and $\mathcal{U}^i$ denote, respectively, the observation and the action spaces of agent i.

The team incurs a cost c, which is a nonnegative continuous and bounded function of the state, observations, and control actions of all agents, that is, $c : \mathcal{X} \times \mathcal{Y}^{1:N} \times \mathcal{U}^{1:N} \to \mathbb{R}^+$. The expected cost functional of the team, denoted by $J : \mathcal{R}^{1:N} \to \mathbb{R}^+$, as a function of behavioral strategies of the agents, is

$$J(\pi^{1:N}) = \int_{\mathcal{X} \times \mathcal{Y}^{1:N}} \int_{\mathcal{U}^{1:N}} c(\mathbf{x}, \mathbf{y}^{1:N}, \mathbf{u}^{1:N}) \prod_{i=1}^{N} \pi^i(d\mathbf{u}^i | \mathbf{y}^i) \mathbb{P}\left\{ d\mathbf{x}, d\mathbf{y}^1, \ldots, d\mathbf{y}^N \right\}.$$

We show that, under certain conditions, there exists a $\pi^{1:N\star} \in \mathcal{R}^{1:N}$ such that $J(\pi^{1:N\star}) = \inf_{\pi^{1:N} \in \mathcal{R}^{1:N}} J(\pi^{1:N})$.

We first provide an outline of our approach to showing the existence of optimal strategies in the static team problem. Consider a sequence $\{\pi_n^{1:N}\}_{n \in \mathbb{N}} \subset \mathcal{R}^{1:N}$ of control strategies of the agents such that $\lim_{n \to \infty} J(\pi_n^{1:N}) = \inf_{\pi^{1:N} \in \mathcal{R}^{1:N}} J(\pi^{1:N})$. There are three issues that need to be resolved. The first is that the sequence of joint measures $\{\prod_{i=1}^{N} \pi_n^i(d\mathbf{u}^i | \mathbf{y}^i) \mathbb{P}\left\{ d\mathbf{x}, d\mathbf{y}^1, \ldots, d\mathbf{y}^N \right\}\}_{n \in \mathbb{N}}$ may not be a weak* convergent sequence. This can be remedied by considering a convergent subsequence of $\{\prod_{i=1}^{N} \pi_n^i(d\mathbf{u}^i | \mathbf{y}^i) \mathbb{P}\left\{ d\mathbf{x}, d\mathbf{y}^1, \ldots, d\mathbf{y}^N \right\}\}_{n \in \mathbb{N}}$. The second problem is to ensure that the limit of the convergent subsequence satisfies the informational constraint. This means that the conditional measure on the action space of agent i given the observation of that agent and the state of the limiting measure must be independent of the state for every $i \in [N]$. The third problem is that if for all $i \in [N]$, $\{\pi_n^i(d\mathbf{u}^i | \mathbf{y}^i) \mathbb{P}\left\{ d\mathbf{x}, d\mathbf{y}^i \right\}\}_{n \in \mathbb{N}}$ converges in the weak* sense to a measure $\pi_0^i(d\mathbf{u}^i | \mathbf{y}^i) \mathbb{P}\left\{ d\mathbf{x}, d\mathbf{y}^i \right\}$ for some $\pi_0^i \in \mathcal{R}^i$, then the expected cost functional J may not satisfy $\lim_{n \to \infty} J(\pi_n^{1:N}) = J(\pi_0^{1:N})$. We overcome all three challenges by employing the following steps:

1. We show that for any $g \in U_b(\mathcal{X} \times \mathcal{Y}^{1:N} \times \mathcal{U}^{1:N})$, $\left\{ \int g \, \pi_n^i(d\mathbf{u}^i | \mathbf{y}^i) \mathbb{P}\left\{ d\mathbf{y}^i | \mathbf{x} \right\} \right\}_{n \in \mathbb{N}}$ is a uniformly equicontinuous and bounded sequence of functions under some assumptions on the conditional measure $\mathbb{P}\left\{ d\mathbf{y}^i | \mathbf{x} \right\}$.
2. In order to satisfy the informational constraint of the limiting measure of any convergent subsequence of the sequence $\{\pi_n^i(d\mathbf{u}^i | \mathbf{y}^i) \mathbb{P}\left\{ d\mathbf{x}, d\mathbf{y}^i \right\}\}_{n \in \mathbb{N}}$, we assume a specific structure on the conditional probability measure $\mathbb{P}\left\{ d\mathbf{x} | \mathbf{y}^i \right\}$.

3. We extract $\{\prod_{i=1}^{N} \pi_{n_k}^i (d\mathbf{u}^i | \mathbf{y}^i) \mathbb{P} \{d\mathbf{x}, d\mathbf{y}^1, \ldots, d\mathbf{y}^N\}\}_{k \in \mathbb{N}}$, a weak* convergent subsequence

$$\left\{ \prod_{i=1}^{N} \pi_{n_k}^i (d\mathbf{u}^i | \mathbf{y}^i) \mathbb{P} \{d\mathbf{x}, d\mathbf{y}^1, \ldots, d\mathbf{y}^N\} \right\}_{k \in \mathbb{N}}$$

of measures such that $\lim_{k \to \infty} J(\pi_{n_k}^{1:N}) = J(\pi_0^{1:N})$.

4. Once we show that there exists a set of behavioral strategies of the agents that achieves the minimum expected cost, we use Blackwell's irrelevant information theorem [37, p. 457] to conclude that there exists a set of deterministic strategies of the agents that achieves the minimum expected cost.

In order to prove the existence result, we require several auxiliary results that are proved in the next subsection.

3.1. Auxiliary results. In this subsection, we state a few lemmas that are needed to prove the existence of optimal strategies in the static team problem formulated above. Unless otherwise stated, $\mathcal{A}$, $\mathcal{B}^i$, $\mathcal{C}$, and $\mathcal{Y}^i$, $i \in [N]$, denote Polish spaces, with generic elements in these spaces denoted, respectively, by $\mathbf{a}$, $\mathbf{b}^i$, $\mathbf{c}$, and $\mathbf{y}^i$. We now introduce a condition on the conditional probability measures, which will be important in proving the auxiliary results.

DEFINITION 3.1 (condition **C1**). *Let* $\mathbf{A}$ *and* $\mathbf{Y}$ *be random variables such that* $\mathbb{P}\{d\mathbf{y}|\mathbf{a}\} = \eta(\mathbf{a}, \mathbf{y})\nu(d\mathbf{y})$ *for some nonnegative measure* $\nu \in ca(\mathcal{Y})$. *We say that the pair* (η, ν) *satisfies condition* **C1** *if and only if*

1. η *is a continuous function of its arguments, that is,* $\eta \in C(\mathcal{A} \times \mathcal{Y})$; *and*
2. *there exists a measurable function* $h_{(\eta,\nu)} : \mathcal{A} \times \mathcal{Y} \to \mathbb{R}^+$ *satisfying*

$$\sup_{\mathbf{a} \in \mathcal{A}} \int h_{(\eta,\nu)}(\mathbf{a}, \mathbf{y})\nu(d\mathbf{y}) < \infty$$

such that for every $\epsilon > 0$, *there exists a* $\delta > 0$ *such that for any* $\mathbf{a}_0 \in \mathcal{A}$ *and for all* $\mathbf{a} \in \mathcal{A}$ *satisfying* $d_{\mathcal{A}}(\mathbf{a}, \mathbf{a}_0) < \delta$, *we have*

$$|\eta(\mathbf{a}, \mathbf{y}) - \eta(\mathbf{a}_0, \mathbf{y})| < \epsilon \, h_{(\eta,\nu)}(\mathbf{a}_0, \mathbf{y}).$$

We call $h_{(\eta,\nu)}$ *the variation control (VC) function of the pair* (η, ν).

If the observation is an additive noise corrupted version of the state, with the noise being Gaussian, then the above condition holds, as noted in the following example.

Example 1. Let $\mathcal{A} = \mathcal{Y} = \mathbb{R}^n$. A sufficient condition for a pair (η, ν) to satisfy condition **C1** is $\mathbf{Y} = \mathbf{A} + \mathbf{W}$ for some Gaussian random vector $\mathbf{W}$ with density function $N(\cdot)$ and a positive definite covariance. In this case, ν is the usual Lebesgue measure on $\mathbb{R}^n$, $\eta(\mathbf{a}, \mathbf{y}) = N(\mathbf{y} - \mathbf{a})$, and the VC function for the pair (η, ν) is

$$h_{(\eta,\nu)}(\mathbf{a}_0, \mathbf{y}) := \max_{\mathbf{a} \in B(\mathbf{a}_0, 1)} \left\| \frac{d\eta}{d\mathbf{a}} \right\|_{\infty},$$

where $B(\mathbf{a}_0, 1)$ is a unit ball around $\mathbf{a}_0$ under ∞ norm. Since $\frac{d\eta}{d\mathbf{a}}$ decays exponentially as $\|\mathbf{y}\|_2 \to \infty$, $h_{(\eta,\nu)}$ has the property that $\int h_{(\eta,\nu)} d\nu < \infty$ and it is a constant function over $\mathcal{A}$.

Now, we make use of the uniform continuity of a function and condition **C1** on the conditional measure to prove the following result.

LEMMA 3.2. *Let* $\nu \in ca(\mathcal{Y})$ *be a nonnegative measure and let* $\mu \in \wp(\mathcal{Y} \times \mathcal{B})$. *Let* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, *and* $\mathbf{Y}$ *be random variables such that* $\mathbb{P}\{d\mathbf{y}|\mathbf{a}\} = \eta(\mathbf{a}, \mathbf{y})\nu(d\mathbf{y})$, *where the*

pair (η, ν) satisfies condition **C1** *with VC function* $h_{(\eta,\nu)}$. *Let* $g : \mathcal{A} \times \mathcal{B} \times \mathcal{C} \to \mathbb{R}$ *be a uniformly continuous and bounded function. Define the map* $f : \mathcal{A} \times \mathcal{C} \to \mathbb{R}$ *by*

$$f_\mu(\mathbf{a}, \mathbf{c}) = \int_\mathcal{B} g(\mathbf{a}, \mathbf{b}, \mathbf{c}) \mu(d\mathbf{b}|\mathbf{y}) \eta(\mathbf{a}, \mathbf{y}) \nu(d\mathbf{y}).$$

Then, $\{f_\mu(\cdot, \cdot)\}_{\mu \in \wp(\mathcal{Y} \times \mathcal{B})}$ *is uniformly equicontinuous and bounded on its domain.*

Proof. See Appendix A. □

We now have a corollary to this result.

COROLLARY 3.3. *Under the same assumptions and notation as in Lemma* 3.2, *for every* $\mu \in \wp(\mathcal{Y} \times \mathcal{B})$, f_μ *is a uniformly continuous function on its domain.*

We now state another lemma, whose proof is similar to that of Lemma 3.2.

LEMMA 3.4. *Let* $\nu^i \in \mathsf{ca}(\mathcal{Y}^i)$ *be a nonnegative measure and* $\mu^i \in \wp(\mathcal{Y}^i \times \mathcal{B}^i)$, $i \in [N]$. *Let* $\mathbf{A}$, $\mathbf{B}^i$, $\mathbf{C}$, *and* $\mathbf{Y}^i$ *be random variables for* $i \in [N]$ *such that* $\mathbb{P}\{d\mathbf{y}^i|\mathbf{a}\} = \eta^i(\mathbf{a}, \mathbf{y}^i) \nu^i(d\mathbf{y}^i)$, *where each pair* (η^i, ν^i) *satisfies condition* **C1** *with VC function* h^i. *Further, assume that random variables* $\mathbf{Y}^1, \dots, \mathbf{Y}^N$ *are independent given* $\mathbf{a}$. *Let* $g \in U_b(\mathcal{A} \times \mathcal{B}^{1:N} \times \mathcal{C})$. *Define the map* $f : \mathcal{A} \times \mathcal{C} \to \mathbb{R}$ *by*

$$f_{\mu^{1:N}}(\mathbf{a}, \mathbf{c}) = \int_{\mathcal{B}^{1:N}} g(\mathbf{a}, \mathbf{b}^{1:N}, \mathbf{c}) \prod_{i=1}^N \mu^i(d\mathbf{b}^i|\mathbf{y}^i) \eta^i(\mathbf{a}, \mathbf{y}^i) \nu^i(d\mathbf{y}^i).$$

Then, $\{f_{\mu^{1:N}}(\cdot, \cdot)\}_{\mu^i \in \wp(\mathcal{Y}^i \times \mathcal{B}^i)}$ *is uniformly equicontinuous and bounded on its domain.*

Proof. See Appendix B. □

The result of Lemma 3.4 allows us to apply the Arzela–Ascoli theorem [1] on compact subsets of the domain to obtain a convergent subsequence that converges to some bounded continuous function pointwise (not in sup norm). We then need the following.

LEMMA 3.5. *Under the same assumptions and notation as in Lemmas* 3.2 *and* 3.4, *let* $\{\mu_n^i\}_{n \in \mathbb{N}} \subset \wp(\mathcal{Y}^i \times \mathcal{B}^i)$ *be an arbitrary sequence of measures for* $i \in [N]$. *For every* $n \in \mathbb{N}$, *define* $f_n := f_{\mu_n^{1:N}}$. *Further, assume that* $\mathcal{A}, \mathcal{C}, \mathcal{Y}^i, \mathcal{B}^i, i \in [N]$, *are all* σ*-compact Polish spaces. If* $\{\zeta_n\}_{n \in \mathbb{N}} \subset \wp(\mathcal{A} \times \mathcal{C})$ *is a weak* convergent sequence of measures converging to* ζ_0, *then there exists a subsequence* $\{n_k\}_{k \in \mathbb{N}}$ *such that*

$$\lim_{k \to \infty} \left| \int_{\mathcal{A} \times \mathcal{C}} f_{n_k} d\zeta_{n_k} - \int_{\mathcal{A} \times \mathcal{C}} f_{n_k} d\zeta_0 \right| = 0.$$

Proof. See Appendix C. □

We have thus stated (and proved) all the major auxiliary results that are needed to establish the existence of optimal strategies. We next prove an additional result, which states that under some sufficient condition, if we take a weak* convergent sequence of measures satisfying a conditional independence property, then the limit also satisfies the conditional independence property. This result is useful to show that the weak* convergent sequence of joint measures over the state, observation, and action of agent i does not converge to a limit in which the control action depends on both the state and the observation.

LEMMA 3.6. *Let* $\{\mu_n\}_{n \in \mathbb{N}} \subset \wp(\mathcal{A} \times \mathcal{B} \times \mathcal{C})$ *be a convergent sequence of measures such that* $\mu_n(d\mathbf{a}, d\mathbf{b}, d\mathbf{c}) = \mu_n(d\mathbf{c}|\mathbf{b}) \zeta(d\mathbf{a}, d\mathbf{b})$, *where* $\zeta \in \wp(\mathcal{A} \times \mathcal{B})$ *with the property that* $\zeta(d\mathbf{a}|\mathbf{b}) = \rho(\mathbf{a}, \mathbf{b}) \nu(d\mathbf{a})$ *for some* $\rho \in C(\mathcal{A} \times \mathcal{B})$ *and nonnegative measure* ν *on* $\mathcal{A}$. *Assume that* (ρ, ν) *satisfies condition* **C1**. *If* $\mu_n \overset{w^*}{\to} \mu_0$ *for some* $\mu_0 \in \wp(\mathcal{A} \times \mathcal{B} \times \mathcal{C})$, *then* $\mu_0(d\mathbf{a}, d\mathbf{b}, d\mathbf{c}) = \mu_0(d\mathbf{c}|\mathbf{b}) \zeta(d\mathbf{a}, d\mathbf{b})$.

Proof. See Appendix D. □

In the next subsection, we turn our attention to proving the existence of optimal strategies for the static team problem considered in this section.

3.2. Existence of optimal strategies. Here, we prove one of the main results of the paper under the following assumption on the probability measure on $\mathcal{X} \times \mathcal{Y}^{1:N}$.

ASSUMPTION 3.1. *The spaces $\mathcal{X}, \mathcal{Y}^i$, and $\mathcal{U}^i$ are σ-compact Polish spaces for all $i \in [N]$. Further, there exist continuous functions η^i and ρ^i, $i \in [N]$, such that $\mathbb{P}\left\{ d\mathbf{y}^{1:N}|\mathbf{x} \right\} = \prod_{i \in [N]} \mathbb{P}\left\{ d\mathbf{y}^i|\mathbf{x} \right\}, \mathbb{P}\left\{ d\mathbf{y}^i|\mathbf{x} \right\} = \eta^i(\mathbf{x}, \mathbf{y}^i)\nu_{\mathcal{Y}^i}(d\mathbf{y}^i)$, and $\mathbb{P}\left\{ d\mathbf{x}|\mathbf{y}^i \right\} = \rho^i(\mathbf{y}^i, \mathbf{x})\nu_{\mathcal{X}}(d\mathbf{x})$, where $\nu_{\mathcal{X}}$ and $\nu_{\mathcal{Y}^i}$ are nonnegative measures on $\mathcal{X}$ and $\mathcal{Y}^i$, respectively. The tuples $(\rho^i, \nu_{\mathcal{X}})$ and $(\eta^i, \nu_{\mathcal{Y}^i})$, $i \in [N]$, satisfy condition* **C1**.

THEOREM 3.7. *Consider team* **ST1**, *satisfying Assumption* 3.1, *where $\mathcal{U}^i, i \in [N]$, need not be compact sets. If for every $i \in [N]$, $\{\lambda_n^i\}_{n \in \mathbb{N}} \subset \wp(\mathcal{U}^i \times \mathcal{Y}^i \times \mathcal{X})$ is a convergent sequence of measures such that $\lambda_n^i(d\mathbf{u}^i, d\mathbf{y}^i, d\mathbf{x}) = \lambda_n^i(d\mathbf{u}^i|\mathbf{y}^i)\mathbb{P}\left\{ d\mathbf{x}, d\mathbf{y}^i \right\}$, converging to some $\lambda_0^i \in \wp(\mathcal{U}^i \times \mathcal{Y}^i \times \mathcal{X})$, then*

$$\left(\prod_{i=1}^{N} \lambda_n^i(d\mathbf{u}^i|\mathbf{y}^i) \right) \mathbb{P}\left\{ d\mathbf{x}, d\mathbf{y}^{1:N} \right\} \xrightarrow{w^*} \left(\prod_{i=1}^{N} \lambda_0^i(d\mathbf{u}^i|\mathbf{y}^i) \right) \mathbb{P}\left\{ d\mathbf{x}, d\mathbf{y}^{1:N} \right\} \text{ as } n \to \infty.$$

Proof. See Appendix E. □

We now list the conditions that we need in order to establish the main result.

ASSUMPTION 3.2.
1. *The cost function $c \in C_b(\mathcal{X} \times \mathcal{Y}^{1:N} \times \mathcal{U}^{1:N})$ is nonnegative.*
2. *The action sets $\mathcal{U}^i, i \in [N]$, of all the agents are compact subsets of Polish spaces. Therefore, $\mathcal{U}^i$ is a σ-compact Polish space for all $i \in [N]$.*
3. *Assumption* 3.1 *holds.*

Remark 3.1. It should also be noted that part 3 of Assumption 3.2 is satisfied if (i) $\mathcal{X}$ and $\mathcal{Y}^i$, $i \in [N]$, are finite dimensional Euclidean spaces, and (ii) the state and observations are jointly Gaussian random variables such that agent i observes a Gaussian noise corrupted version of the state $\mathbf{X}$.

The following theorem states that any team problem that satisfies the assumptions made above admits a team-optimal solution.

THEOREM 3.8. *Every static team problem satisfying Assumption* 3.2 *admits a team-optimal solution in deterministic strategies.*

Proof. Let $\{\pi_n^{1:N}\}_{n \in \mathbb{N}} \subset \mathcal{R}^{1:N}$ be a sequence of strategy profiles of the agents such that $J(\pi_n^{1:N}) < \inf_{\pi^{1:N} \in \mathcal{R}^{1:N}} J(\pi^{1:N}) + \frac{1}{n}$. We next show that there exists a convergent subsequence of this sequence $\{\pi_n^{1:N}\}_{n \in \mathbb{N}}$ such that the limiting behavioral strategies of the agents achieve the infimum of the expected cost functional. We organize the proof into four steps.

Step 1 (*tightness*). Since $\mathcal{X}$ and $\mathcal{Y}^i$ are Polish spaces, $\mathbb{P}\left\{ d\mathbf{x}, d\mathbf{y}^i \right\}$ is a tight measure.[2] Since $\mathcal{U}^i$ is compact, the set of measures $\{\pi^i(d\mathbf{u}^i|\mathbf{y}^i)\mathbb{P}\left\{ d\mathbf{x}, d\mathbf{y}^i \right\}\}_{\pi^i \in \mathcal{R}^i}$ is tight for all $i \in [N]$. Define a measure λ_n^i as $\lambda_n^i(d\mathbf{u}^i, d\mathbf{y}^i, d\mathbf{x}) := \pi_n^i(d\mathbf{u}^i|\mathbf{y}^i)\mathbb{P}\left\{ d\mathbf{x}, d\mathbf{y}^i \right\}$ for $n \in \mathbb{N}$, $i \in [N]$.

Step 2 (*extracting convergent subsequence*). Recall that every sequence of tight measures has a convergent subsequence by Prohorov's theorem[3] [14]. Thus, $\{\lambda_n^1\}_{n \in \mathbb{N}}$

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[2]Let $\mathcal{A}$ be a Polish space. A set $\mathcal{M} \subset \wp(\mathcal{A})$ is said to be tight if for every $\epsilon > 0$, there exists a compact set $\mathcal{K}_\epsilon \subset \mathcal{A}$ such that $\mu(\mathcal{A} \setminus \mathcal{K}_\epsilon) < \epsilon$ for all $\mu \in \mathcal{M}$. Every measure on a Polish space is tight [14].

[3]Prohorov's theorem states that a set of measures over a Polish space is weak* precompact if and only if it is tight.

must have a convergent subsequence, say $\{\lambda_{n_k}^1\}_{k\in\mathbb{N}}$. Similarly, $\{\lambda_{n_k}^2\}_{k\in\mathbb{N}}$ must have a convergent subsequence. Since there is only a finite number (N) of agents, we continue this process of extracting convergent subsequences of every sequence of measures to get a convergent subsequence of a set of measures $\{\lambda_{n_l}^1,\ldots,\lambda_{n_l}^N\}_{l\in\mathbb{N}}$ such that $\lambda_{n_l}^i \overset{w^*}{\rightharpoonup} \lambda_0^i$ as $l\to\infty$ for all $i\in[N]$ for some set of measures $\{\lambda_0^1,\ldots,\lambda_0^N\}$. Recall the result of Lemma 3.6, which implies that $\lambda_0^i(d\mathbf{u}^i,d\mathbf{y}^i,d\mathbf{x}) = \lambda_0^i(d\mathbf{u}^i|\mathbf{y}^i)\mathbb{P}\{d\mathbf{x},d\mathbf{y}^i\}$. Define $\pi_0^i(d\mathbf{u}^i|\mathbf{y}^i) := \lambda_0^i(d\mathbf{u}^i|\mathbf{y}^i)$ for all $i\in[N]$ and $\pi_0^{1:N} := \{\pi_0^1,\ldots,\pi_0^N\}$.

Step 3 *(limit achieves infimum).* The result of Theorem 3.7 implies that

$$\lim_{l\to\infty} J(\pi_{n_l}^{1:N}) = J(\pi_0^{1:N}) = \inf_{\pi^{1:N}\in\mathcal{R}^{1:N}} J(\pi^{1:N}).$$

Step 4 *(applying a result on irrelevant information due to Blackwell)* [8, 7]. Now, using Blackwell's irrelevant information theorem (see, e.g., [37, p. 457]), we conclude that for fixed optimal behavioral strategies π_0^{-i} of all agents other than i, there exists a deterministic strategy of agent i, say γ_0^i, such that $J(\pi_0^{1:N}) \geq J(\gamma_0^i,\pi_0^{-i})$. We continue this process sequentially to conclude that there exists $\gamma_0^{1:N}$ such that

$$J(\pi_0^{1:N}) \geq J(\gamma_0^N,\pi_0^{1:N-1}) \geq \cdots \geq J(\gamma_0^{1:N}).$$

Since $\inf_{\mathcal{D}^{1:N}} J(\gamma^{1:N}) \geq J(\pi_0^{1:N})$ (because the set of randomized strategies of an agent subsumes its set of deterministic strategies), we conclude that the above inequalities hold with equality. Thus, all agents' strategies can be restricted, without any loss of generality, to deterministic ones. This implies that the static team admits an optimal solution in the class of deterministic strategies of the agents, which completes the proof of the theorem. □

Remark 3.2. It must be noted that for the existence result proven above, we do not require the state space $\mathcal{X}$ and observation spaces $\mathcal{Y}^i, i\in[N]$, to be compact.

We have an immediate corollary.

COROLLARY 3.9. *Assume that every agent $i\in[N]$ observes $\mathbf{Y}^i$, which is independent of the observations of all other agents. If the cost function is continuous in its arguments (observations and actions of the agents) and bounded, and action spaces of the agents are compact, then the static team with independent observations admits an optimal solution in deterministic strategies.*

3.3. Static team ST1 with degraded information. In Assumption 3.1, we had conditional independence of observations given the state, which we relax in this subsection. For simplicity, we consider a two-agent static team problem, where the observation of agent 2 is a noise corrupted version of the observation of agent 1. We further invoke the assumption below. Lemma 3.10 and Theorem 3.11 below and the main idea of the proof can readily be extended to multiagent scenarios.

ASSUMPTION 3.3. *Consider team* **ST1** *in which agent 2 observes a degraded version of agent 1's observation. The spaces $\mathcal{X}$, $\mathcal{Y}^i$, and $\mathcal{U}^i$ are σ-compact Polish spaces for $i\in\{1,2\}$. There exist continuous functions η^i and ρ^i, $i\in\{1,2\}$, such that*

$$\mathbb{P}\{d\mathbf{y}^2,d\mathbf{y}^1|\mathbf{x}\} = \mathbb{P}\{d\mathbf{y}^2|\mathbf{y}^1\}\mathbb{P}\{d\mathbf{y}^1|\mathbf{x}\}, \quad \mathbb{P}\{d\mathbf{y}^2|\mathbf{y}^1\} = \eta^2(\mathbf{y}^1,\mathbf{y}^2)\nu_{\mathcal{Y}^2}(d\mathbf{y}^2),$$
$$\mathbb{P}\{d\mathbf{y}^1|\mathbf{x},\mathbf{y}^2\} = \eta^1(\mathbf{x},\mathbf{y}^2,\mathbf{y}^1)\nu_{\mathcal{Y}^1}(d\mathbf{y}^1), \qquad \mathbb{P}\{d\mathbf{x}|\mathbf{y}^1\} = \rho^1(\mathbf{y}^1,\mathbf{x})\nu_{\mathcal{X}}(d\mathbf{x}),$$
$$\mathbb{P}\{d\mathbf{y}^1,d\mathbf{x}|\mathbf{y}^2\} = \rho^2(\mathbf{y}^2,\mathbf{y}^1,\mathbf{x})\nu_{\mathcal{Y}^1}(d\mathbf{y}^1)\nu_{\mathcal{X}}(d\mathbf{x}),$$

where $\nu_{\mathcal{X}}$ and $\nu_{\mathcal{Y}^i}$ are nonnegative measures on $\mathcal{X}$ and $\mathcal{Y}^i$, respectively. The tuples $(\rho^1,\nu_{\mathcal{X}})$, $(\rho^2,\nu_{\mathcal{X}}\times\nu_{\mathcal{Y}^1})$, and $(\eta^i,\nu_{\mathcal{Y}^i})$, $i\in\{1,2\}$, satisfy condition **C1**.

We now use the auxiliary results in the previous subsection to prove the following.

LEMMA 3.10. *Consider team* **ST1** *satisfying Assumption* 3.3, *where* $\mathcal{U}^i, i \in \{1,2\}$, *need not be compact sets. Let* $\mathcal{X}, \mathcal{Y}^i, \mathcal{U}^i, i \in \{1,2\}$, *be* σ-*compact Polish spaces. For every* $i \in \{1,2\}$, *let* $\{\lambda_n^1\}_{n\in\mathbb{N}} \subset \wp(\mathcal{U}^i \times \mathcal{Y}^1 \times \mathcal{X})$ *and* $\{\lambda_n^2\}_{n\in\mathbb{N}} \subset \wp(\mathcal{U}^i \times \mathcal{Y}^1 \times \mathcal{Y}^2 \times \mathcal{X})$ *be convergent sequences of measures such that* $\lambda_n^1(d\mathbf{u}^1, d\mathbf{y}^1, d\mathbf{x}) = \lambda_n^1(d\mathbf{u}^1|\mathbf{y}^1)\mathbb{P}\{d\mathbf{x}, d\mathbf{y}^1\}$ *and* $\lambda_n^2(d\mathbf{u}^2, d\mathbf{y}^{1:2}, d\mathbf{x}) = \lambda_n^2(d\mathbf{u}^2|\mathbf{y}^2)\mathbb{P}\{d\mathbf{x}, d\mathbf{y}^{1:2}\}$, *converging to some* $\lambda_0^1 \in \wp(\mathcal{U}^1 \times \mathcal{Y}^1 \times \mathcal{X})$ *and* $\lambda_0^2 \in \wp(\mathcal{U}^2 \times \mathcal{Y}^1 \times \mathcal{Y}^2 \times \mathcal{X})$, *respectively. Then,*

$$\left(\prod_{i=1}^{2}\lambda_n^i(d\mathbf{u}^i|\mathbf{y}^i)\right)\mathbb{P}\{d\mathbf{x}, d\mathbf{y}^{1:2}\} \xrightarrow{w^*} \left(\prod_{i=1}^{2}\lambda_0^i(d\mathbf{u}^i|\mathbf{y}^i)\right)\mathbb{P}\{d\mathbf{x}, d\mathbf{y}^{1:2}\} \quad as\ n \to \infty.$$

Proof. See Appendix F. □

We can now show the existence of optimal strategies for a static team **ST1** satisfying Assumption 3.2 (1,2) and Assumption 3.3.

THEOREM 3.11. *Any two-agent static team* **ST1** *with degraded information satisfying Assumption* 3.2 (1,2) *and Assumption* 3.3 *admits a team-optimal solution, which is in the class of deterministic strategies of the agents.*

Proof. The proof follows by mimicking the steps of the proof of Theorem 3.8 and using Lemma 3.10. □

In the above theorem, we showed that the assumption of conditional independence of observations given the state is not needed for the existence of optimal strategies in team problems; we considered a case where the observation of one agent is a degraded version of the observation of another agent and showed that optimal strategies exist under certain assumptions.

We note that our setting does not cover teams with observation sharing information structures, because the technique we employed for proving Theorem 3.8 does not readily carry over to such teams. In particular, if the agents share their observations in a certain manner, then we cannot show the equicontinuity result of Lemma 3.4, which is used to prove Theorem 3.7. Recall that Theorem 3.7 is crucial for the proof of Theorem 3.8. Consequently, the proof technique we adopted above cannot be carried forward to prove the existence of team-optimal solutions in static teams with observation sharing information pattern. In fact, we provide an example in [17, section 9.3, p. 141], in which a counterpart of Theorem 3.7 does not hold. However, for a class of teams with observation sharing information pattern, it is possible to use other techniques, such as dynamic programming or viewing the decision makers with common information as a single decision maker. This technique has been used in [17, Chapter 9, p. 130] for a two-agent team in which only one agent shares observation with the other agent. A systematic analysis of this technique for general teams is left for future work.

4. Existence of optimal solution in ST2. Here, we consider the static team problem in which the cost function is nonnegative and continuous, but it may be unbounded, and the action sets may be noncompact. We build on the results proved in the previous section to show the existence of optimal strategies.

In the next subsection, we use the result from Theorem 3.7 to investigate the properties of the expected cost functional, as a function of the behavioral strategies of the agents, of the team problem with unbounded cost and noncompact action spaces.

4.1. Properties of the expected cost functional. Our first result uses Theorem 3.7 to prove a property of expected cost functional of team **ST2**.

THEOREM 4.1. *Recall that* $c : \mathcal{X} \times \mathcal{Y}^{1:N} \times \mathcal{U}^{1:N} \to \mathbb{R}^+$ *is a nonnegative continuous function. Under the same assumptions and notation as in Theorem* 3.7, *we get*

$$\liminf_{n \to \infty} \int_{\mathcal{X} \times \mathcal{Y}^{1:N} \times \mathcal{U}^{1:N}} c(\mathbf{x}, \mathbf{y}^{1:N}, \mathbf{u}^{1:N}) \prod_{i=1}^{N} \lambda_n^i (d\mathbf{u}^i | \mathbf{y}^i) \mathbb{P}\left\{d\mathbf{x}, d\mathbf{y}^{1:N}\right\}$$

$$\geq \int_{\mathcal{X} \times \mathcal{Y}^{1:N} \times \mathcal{U}^{1:N}} c(\mathbf{x}, \mathbf{y}^{1:N}, \mathbf{u}^{1:N}) \prod_{i=1}^{N} \lambda_0^i (d\mathbf{u}^i | \mathbf{y}^i) \mathbb{P}\left\{d\mathbf{x}, d\mathbf{y}^{1:N}\right\}.$$

Proof. The statement follows from Theorem 3.7 and [31, Lemma 4.3, p. 43]. $\square$

The theorem above says that the expected cost functional of the agents in the team is lower-semicontinuous on the space of joint measures over state, observation, and action spaces. Since the action sets are noncompact, the set of joint measures over observation and action spaces of each agent is noncompact, and we cannot readily use Weierstrass theorem like results to prove the existence of an optimal solution.

We address this issue in the next subsection. In particular, if the cost function has some stronger (coercivity like) property, then using Assumption 2.2 and Markov's inequality, we can restrict the search of optimal strategies of the agents to compact sets of joint measures over observation and action spaces of the agents.

4.2. Compactness of a set of probability measures. In this subsection, we identify a sufficient condition for a set of measures to be precompact in the weak* topology. We use this result later to show that the search for optimal behavioral strategies in the team problem can be restricted to a weak* precompact space.

Hereafter, we will use $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ to denote arbitrary Polish spaces. We now define a class of functions and study an important result involving functions in this class.

DEFINITION 4.2 (class $\mathrm{IC}(\mathcal{A}, \mathcal{B})$). *A nonnegative measurable function* $\phi : \mathcal{A} \times \mathcal{B} \times \mathcal{C} \to \mathbb{R}$ *is in class* $\mathrm{IC}(\mathcal{A}, \mathcal{B})$ *if* ϕ *satisfies any one of the following two conditions:*

1. *For every* $M > 0$ *and for every compact set* $\mathcal{K} \subset \mathcal{A}$, *there exists a compact set* $\mathcal{L} \subset \mathcal{B}$ *such that* $\inf_{\mathcal{K} \times \mathcal{L}^\complement \times \mathcal{C}} \phi(\mathbf{a}, \mathbf{b}, \mathbf{c}) \geq M$.
2. *For every* $M > 0$ *and every point* $\mathbf{a} \in \mathcal{A}$, *there exists an open neighborhood* $\mathcal{O} \subset \mathcal{A}$ *of the point* $\mathbf{a} \in \mathcal{A}$ *and a compact set* $\mathcal{L} \subset \mathcal{B}$ *such that* $\inf_{\mathcal{O} \times \mathcal{L}^\complement \times \mathcal{C}} \phi(\mathbf{a}, \mathbf{b}, \mathbf{c}) \geq M$.

We can have $\mathcal{C} = \emptyset$.

A large class of team problems has cost functions that belong to the class of functions defined above, where $\mathcal{A}$ is the space of primitive random variables and $\mathcal{B}$ is an action space of some agent. This class of functions is therefore an important one, and we will exploit this property of cost function to show the existence of an optimal solution in a team. We first identify a few examples of functions in class $\mathrm{IC}(\mathcal{A}, \mathcal{B})$.

Example 2. Let $\mathcal{A} = \mathcal{B} = \mathcal{C} = \mathbb{R}^n$, and define $\phi_1(\mathbf{a}, \mathbf{b}, \mathbf{c}) := \|\mathbf{b} - \mathbf{a}\| + \|\mathbf{c}\|$ and $\phi_2(\mathbf{a}, \mathbf{b}, \mathbf{c}) := \|\mathbf{b} - \mathbf{a}\|^2$. Then, ϕ_1 and ϕ_2 are in class $\mathrm{IC}(\mathcal{A}, \mathcal{B})$. Any nonnegative continuous and increasing function on $\mathbb{R}$ composed with ϕ_1 or ϕ_2 is also in class $\mathrm{IC}(\mathcal{A}, \mathcal{B})$. For example, $\exp(\phi_1(\mathbf{a}, \mathbf{b}, \mathbf{c}))$ and $\exp(\phi_2(\mathbf{a}, \mathbf{b}, \mathbf{c}))$ are in class $\mathrm{IC}(\mathcal{A}, \mathcal{B})$.

Our next result gives a sufficient condition for a set of measures to be tight, which uses the class of functions introduced in Definition 4.2.

LEMMA 4.3 (tightness of a set of measures). *Let* $\phi : \mathcal{A} \times \mathcal{B} \times \mathcal{C} \to \mathbb{R}$ *be a nonnegative measurable function in the class* $\mathrm{IC}(\mathcal{A}, \mathcal{B})$. *Fix* k *to be a nonnegative*

real number and let $\mathcal{N} \subset \wp(\mathcal{A})$ be a weak* compact set of measures. Define $\mathcal{M} \subset \wp(\mathcal{A} \times \mathcal{B} \times \mathcal{C})$ as follows:

$$\mathcal{M} = \left\{ \mu \in \wp(\mathcal{A} \times \mathcal{B} \times \mathcal{C}) : Pr_{\#}^{\mathcal{A}}\mu \in \mathcal{N} \text{ and } \int \phi \, d\mu \leq k \right\}.$$

Then, $Pr_{\#}^{\mathcal{A} \times \mathcal{B}}\mathcal{M}$ is a tight set of measures. Furthermore, if ϕ is lower-semicontinuous, then $Pr_{\#}^{\mathcal{A} \times \mathcal{B}}\mathcal{M}$ is weak* compact.

Proof. See Appendix G. $\quad\square$

Now that we have a sufficient condition on when a set of measures is tight, we can look at the original static team problem in the next subsection.

4.3. Existence of optimal strategies. We need the following assumption.

ASSUMPTION 4.1. *The cost function $c : \mathcal{X} \times \mathcal{Y}^{1:N} \times \mathcal{U}^{1:N} \to \mathbb{R}^+$ is a nonnegative continuous function in class $\mathrm{IC}(\mathcal{X} \times \mathcal{Y}^{1:N}, \mathcal{U}^i)$ for every $i \in [N]$.*

It should be noted that the condition in Assumption 4.1 is not dependent on the control strategies that the agents choose. The following lemma identifies a set of tight measures using Lemma 4.3, with the property that any expected cost below a certain threshold is either achieved by measures in that set or cannot be achieved.

LEMMA 4.4. *Assume that team **ST2** satisfies Assumptions 2.2 and 3.1. Let $\tilde{\pi}^{1:N} \in \mathcal{R}^{1:N}$ be the set of behavioral strategies of the agents which results in finite expected cost to the team. Consider sets $\mathcal{P}^i \subset \mathcal{R}^i, i \in [N]$ such that there exists a set of behavioral strategies $\pi^{1:N} \in \mathcal{P}^{1:N}$ satisfying $J(\pi^{1:N}) \leq J(\tilde{\pi}^{1:N})$. For $i \in [N]$, define*

$$\mathcal{M}^i := \bigcup_{\pi^i \in \mathcal{P}^i} \left\{ \lambda^i \in \wp(\mathcal{X} \times \mathcal{Y}^i \times \mathcal{U}^i) : \lambda^i(d\mathbf{u}^i, d\mathbf{y}^i, d\mathbf{x}) = \pi^i(d\mathbf{u}^i|\mathbf{y}^i)\mathbb{P}\left\{d\mathbf{y}^i, d\mathbf{x}\right\} \right\}.$$

If the cost function of the team satisfies Assumption 4.1, then $\mathcal{M}^i \subset \wp(\mathcal{X} \times \mathcal{Y}^i \times \mathcal{U}^i)$ is a tight set of measures for all $i \in [N]$.

Proof. The statement of the lemma readily follows from Lemma 4.3. Define $\mathcal{M} := \{\mu \in \wp(\mathcal{X} \times \mathcal{Y}^{1:N} \times \mathcal{U}^{1:N}) : \int c \, d\mu \leq J(\tilde{\pi}^{1:N})\}$. For every $i \in [N]$, notice that any $\lambda^i \in \mathcal{M}^i$ satisfies $\lambda^i = Pr_{\#}^{\mathcal{X} \times \mathcal{Y}^i \times \mathcal{U}^i}\mu$ for some $\mu \in \mathcal{M}$. Since c is in class $\mathrm{IC}(\mathcal{X} \times \mathcal{Y}^{1:N}, \mathcal{U}^i)$, by Lemma 4.3, $\mathcal{M}^i$ is tight. $\quad\square$

Thus, we have identified precompact sets of joint measures $\mathcal{M}^i, i \in [N]$, which include the optimal joint measures, if they exist. This brings us to the following main result of the section.

THEOREM 4.5. *Assume that the cost function of team **ST2** satisfies Assumption 4.1. If Assumptions 2.2 and 3.1 hold, then team **ST2** admits an optimal solution in deterministic strategies.*

Proof. Let $\tilde{\pi}^{1:N} \in \mathcal{R}^{1:N}$ be the set of behavioral strategies which results in finite expected cost to the team. From Lemma 4.4, we know that there exist tight sets of measures $\mathcal{M}^i \subset \wp(\mathcal{X} \times \mathcal{Y}^i \times \mathcal{U}^i), i \in [N]$, that contain the optimal joint measures, if they exist. Consider a sequence of behavioral strategies $\{\pi_n^{1:N}\}_{n \in \mathbb{N}} \subset \mathcal{R}^{1:N}$ that satisfies $J(\pi_n^{1:N}) \leq J(\tilde{\pi}^{1:N})$ and $\lim_{n \to \infty} J(\pi_n^{1:N}) = \inf_{\pi^{1:N} \in \mathcal{R}^{1:N}} J(\pi^{1:N})$.

Define $\lambda_n^i(d\mathbf{u}^i, d\mathbf{y}^i, d\mathbf{x}) := \pi_n^i(d\mathbf{u}^i|\mathbf{y}^i)\mathbb{P}\left\{d\mathbf{y}^i, d\mathbf{x}\right\}$ for $i \in [N]$ and $n \in \mathbb{N}$, and notice that $\{\lambda_n^i\}_{n \in \mathbb{N}} \subset \mathcal{M}^i$. Since $\{\lambda_n^i\}_{n \in \mathbb{N}}$ is a tight sequence of measures, we know that there exists a weak* convergent subsequence of measures. For every $i \in [N]$, let $\{\lambda_{n_k}^i\}_{k \in \mathbb{N}}$ be the weak* convergent subsequence of measures converging to λ_0^i. From Lemma 3.6, we know that $\lambda_0^i(d\mathbf{u}^i, d\mathbf{y}^i, d\mathbf{x}) = \lambda_0^i(d\mathbf{u}^i|\mathbf{y}^i)\mathbb{P}\left\{d\mathbf{y}^i, d\mathbf{x}\right\}$ for all $i \in [N]$, which means that the conditional independence property is retained in the limit. Let

$\pi_0^i \in \mathcal{R}^i$ be such that $\pi_0^i(d\mathbf{u}^i|\mathbf{y}^i) = \lambda_0^i(d\mathbf{u}^i|\mathbf{y}^i)$. From the result of Theorem 4.1, we conclude that $\liminf_{k\to\infty} J(\pi_{n_k}^{1:N}) \geq J(\pi_0^{1:N})$. Thus, π_0^i is the optimal behavioral strategy of agent i. Moreover, applying Blackwell's irrelevant information theorem [37, p. 457], there exists a set of deterministic strategies which achieves the same cost as the one achieved using optimal behavioral strategies of the agents. $\quad\square$

COROLLARY 4.6. *Consider a two-agent static team* **ST2** *with degraded information. Assume that the cost function of team* **ST2** *satisfies Assumption* 4.1. *If Assumptions* 2.2 *and* 3.3 *hold, then team* **ST2** *admits an optimal solution in deterministic strategies.*

Proof. The proof follows from arguments similar to those used in the proof of Theorems 3.11 and 4.5. $\quad\square$

COROLLARY 4.7. *Assume that the cost function of team* **ST2** *is continuous in its arguments and the action spaces of the agents are compact subsets of Polish spaces. Furthermore, assume that team* **ST2** *satisfies Assumption* 2.2. *If either Assumptions* 3.1 *or* 3.3 *holds for team* **ST2***, then the team admits an optimal solution in deterministic strategies.*

Proof. If the action spaces of the agents are compact, then Assumption 4.1 on the cost function holds automatically. Then, we apply the result of Theorem 4.5 to establish the statement. $\quad\square$

In the next section, we use Witsenhausen's static reduction technique to convert a class of dynamic team problems to static teams with independent observations, and then apply the result proved in this section to conclude the existence of optimal strategies in that class of dynamic team problems.

5. Dynamic teams. We now turn our attention to the dynamic team formulated in section 2. It was shown in [32] that a large class of N-agent T-time step dynamic stochastic control problems with certain information structures can be equivalently written as NT-agent static optimization problems. In order to define the equivalent static problem, we introduce the following notation:

$$\Omega_0 = \mathcal{X}_1 \times \mathcal{W}_{1:T}^0, \qquad \Omega_t^i = \mathcal{W}_t^i, \qquad i \in [N], \ t \in [T].$$

We let ω_0 and ω_t^i denote generic elements of Ω_0 and Ω_t^i, respectively. Note that Ω_0 and Ω_t^i are Polish spaces, endowed with the probability measures ξ_{Ω_0} and $\xi_{\Omega_t^i}$, respectively, which are defined as

$$\xi_{\Omega_0} := \xi_{\mathcal{X}_1}\xi_{\mathcal{W}_1^0}\cdots\xi_{\mathcal{W}_T^0}, \qquad \xi_{\Omega_t^i} := \xi_{\mathcal{W}_t^i}, \quad i \in [N], \ t \in [T].$$

With this notation, the cost function of the team is written as $c : \Omega_0 \times \mathcal{Y}_{1:T}^{1:N} \times \mathcal{U}_{1:T}^{1:N} \to \mathbb{R}^+$, and we recall that it is continuous. Also recall that agent i at time t observes $\mathbf{Y}_t^i$ only, that is, its information set is a singleton.

Now, using the static reduction argument, we can transform the original problem into a static team problem with a different cost function. Toward this end, let us rewrite the observations of the agents as

(5.1) $$\mathbf{y}_t^i = h_t^i(\omega_0, \omega_t^i, \mathbf{u}_{1:t-1}^{1:N}),$$

where h_t^i is nonanticipative, that is, h_t^i does not depend on $\mathbf{w}_{t:T}^0$. Note that due to Assumption 2.1, the functions $h_t^i, i \in [N], t \in [T]$, are continuous maps of their arguments. We now make the following assumption.

ASSUMPTION 5.1. *For every $(i,t) \in [N] \times [T]$, there exists a probability measure $\nu_t^i \in \wp(\mathcal{Y}_t^i)$ and a continuous function $\varphi_t^i : \mathcal{Y}_t^i \times \Omega_0 \times \mathcal{U}_{1:t-1}^{1:N} \to \mathbb{R}^+$ such that*

$$\mathbb{P}\left\{\mathbf{y}_t^i \in \mathcal{Y} | \omega_0, \mathbf{u}_{1:t-1}^{1:N}\right\} = \int_{\mathcal{Y}} \varphi_t^i(\mathbf{y}_t^i; \omega_0, \mathbf{u}_{1:t-1}^{1:N})\nu_t^i(d\mathbf{y}_t^i) \quad \text{for all } \mathcal{Y} \in \mathfrak{B}(\mathcal{Y}_t^i).$$

Define $\varphi : \Omega_0 \times \mathcal{Y}_{1:T}^{1:N} \times \mathcal{U}_{1:T}^{1:N} \to \mathbb{R}^+$ as $\varphi(\omega_0, \mathbf{y}_{1:T}^{1:N}, \mathbf{u}_{1:T}^{1:N}) = \prod_{i=1}^N \prod_{t=1}^T \varphi_t^i(\mathbf{y}_t^i; \omega_0, \mathbf{u}_{1:t-1}^{1:N})$. By definition, φ is a continuous function of its arguments.

In the next lemma, we state a sufficient condition on the mapping h_t^i and the noise statistics $\xi_{\Omega_t^i}$ such that the above assumption is satisfied.

LEMMA 5.1. *Assume that all state, action, observation, and noise spaces are Euclidean spaces of appropriate dimensions. For all $i \in [N], t \in [T]$, let $\xi_{\Omega_t^i}$ admit a zero-mean Gaussian density function η_t^i with positive-definite covariance and*

$$h_t^i(\omega_0, \omega_t^i, \mathbf{u}_{1:t-1}^{1:N}) := \check{h}_t^i(\omega_0, \mathbf{u}_{1:t-1}^{1:N}) + \omega_t^i,$$

where $\check{h}_t^i$ is continuous. Then, Assumption 5.1 holds for the dynamic team.

Proof. For every $i \in [N]$ and $t \in [T]$, define φ_t^i and ν_t^i as

$$\varphi_t^i(\mathbf{y}_t^i; \omega_0, \mathbf{u}_{1:t-1}^{1:N}) := \frac{\eta_t^i(\mathbf{y}_t^i - \check{h}_t^i(\omega_0, \mathbf{u}_{1:t-1}^{1:N}))}{\eta_t^i(\mathbf{y}_t^i)}, \qquad \nu_t^i(d\mathbf{y}_t^i) = \eta_t^i(\mathbf{y}_t^i)d\mathbf{y}_t^i.$$

Thus, φ_t^i is a nonnegative continuous map, which establishes the statement. ☐

We now define the reduced static team corresponding to the dynamic team described above.

DEFINITION 5.2 (reduced static team problem). *Consider the NT-agent static team problem with the agents indexed as (i,t). Agent (i,t) observes a random variable $\mathbf{Y}_t^i$ with probability measure ν_t^i, which is independent of observations of all other agents. Agent (i,t), based on the realization $\mathbf{y}_t^i$ of its observation, chooses a control action $\mathbf{u}_t^i$. The cost function for the team is given by $c(\omega_0, \mathbf{y}_{1:T}^{1:N}, \mathbf{u}_{1:T}^{1:N})\varphi(\omega_0, \mathbf{y}_{1:T}^{1:N}, \mathbf{u}_{1:T}^{1:N})$. We call the static team thus defined the reduced static team (RST).*

We now recall the following result from [32], which shows that any dynamic team and its corresponding RST are equivalent optimization problems over the same space of strategies of the agents.

THEOREM 5.3 (see [32]). *Let $J : \mathcal{R}_{1:T}^{1:N} \to \mathbb{R}_+$ be the expected cost functional of the dynamic team, and let $J_{RST} : \mathcal{R}_{1:T}^{1:N} \to \mathbb{R}_+$ be the expected cost functional of the corresponding RST, defined as*

$$J_{RST}(\pi_{1:T}^{1:N}) = \int c\,\varphi \prod_{i=1}^N \prod_{t=1}^T \left(\pi_t^i(d\mathbf{u}_t^i | \mathbf{y}_t^i)\nu_t^i(d\mathbf{y}_t^i)\right) \mathbb{P}\left\{d\omega_0\right\}, \quad \pi_{1:T}^{1:N} \in \mathcal{R}_{1:T}^{1:N}.$$

Then, for any $\pi_{1:T}^{1:N} \in \mathcal{R}_{1:T}^{1:N}$, we have $J(\pi_{1:T}^{1:N}) = J_{RST}(\pi_{1:T}^{1:N})$.

It should be noted that for any dynamic team that admits a static reduction, the corresponding RST may not satisfy the hypotheses of Theorem 4.5. Thus, the results we proved for static teams cannot be applied to conclude the existence of a solution to a dynamic team problem. We illustrate the difficulty in using such an approach in the following example.

Witsenhausen's counterexample. Witsenhausen's counterexample is a two-agent dynamic LQG team problem, first studied in [33]. The first agent observes a mean-zero unit variance Gaussian random variable y_1 and decides on a real number

U_1. The second agent observes $Y_2 := U_1 + W_2$, where W_2 is a mean-zero Gaussian noise with unit variance, and decides on another real number U_2. The setup is also depicted in Figure 1 section 7. The behavioral strategy space of agent i is $\mathcal{R}_i$, $i = 1, 2$. The cost function of the team is $c_D(y_1, u_1, u_2) = (u_1 - y_1)^2 + (u_2 - u_1)^2$.

It is well known that the dynamic team admits an optimal solution [33]. It can be reduced to a static team using our Lemma 5.1 [32] as follows:

$$
\begin{aligned}
J(\pi_{1:2}) &= \int c_D(y_1, u_{1:2}) \mathbb{P}^{\pi_1,\pi_2}\{du_2, dy_2, du_1, dy_1\} \\
&= \int c_D(y_1, u_{1:2}) \pi_2(du_2|y_2) \mathbb{P}\{dy_2|u_1, y_1\} \pi_1(du_1|y_1) N_{y_1}(y_1) dy_1 \\
&= \int c_D(y_1, u_{1:2}) \pi_2(du_2|y_2) N_{w_2}(y_2 - u_1) dy_2 \pi_1(du_1|y_1) N_{y_1}(y_1) dy_1 \\
&= \int \left(c_D(y_1, u_{1:2}) \frac{N_{w_2}(y_2 - u_1)}{N_{w_2}(y_2)} \right) \pi_2(du_2|y_2) N_{w_2}(y_2) dy_2 \pi_1(du_1|y_1) N_{y_1}(y_1) dy_1,
\end{aligned}
$$

where N_{y_1} and N_{w_2} are the Gaussian density functions (mean-zero, unit variance) of the distributions of Y_1 and W_2, respectively, and $\mathbb{P}^{\pi_1,\pi_2}$ is used to denote the dependence of the probability measure over all random variables due to the choice of randomized strategies (π_1, π_2). In the corresponding RST problem, each agent observes a mean-zero unit variance Gaussian random variable that is independent of the observation of the other agent. The cost function for the RST is

$$
\begin{aligned}
c_S(y_1, u_1, u_2) &= c_D(y_1, u_1, u_2) \frac{N_{w_2}(y_2 - u_1)}{N_{w_2}(y_2)} \\
&= \left((u_1 - y_1)^2 + (u_2 - u_1)^2 \right) \exp\left(\frac{-u_1^2 + 2y_2 u_1}{2} \right).
\end{aligned}
$$

(5.2)

The cost function for the dynamic team c_D is in classes $\mathrm{IC}(\mathcal{Y}_1, \mathcal{U}_1)$ and $\mathrm{IC}(\mathcal{Y}_1 \times \mathcal{Y}_2 \times \mathcal{U}_1, \mathcal{U}_2)$. The cost function c_S for the corresponding RST is in class $\mathrm{IC}(\mathcal{Y}_1 \times \mathcal{Y}_2 \times \mathcal{U}_1, \mathcal{U}_2)$ (which follows from Lemma 5.5, to be introduced and proved later). However, c_S is not in class $\mathrm{IC}(\mathcal{Y}_1, \mathcal{U}_1)$ because as $|u_1| \to \infty$, the cost goes to zero for any fixed value of y_1. Therefore, the result of Theorem 4.5 is not applicable to the RST problem.

The above example illustrates that the results we obtained for the static team problems in sections 3 and 4 are not readily applicable to all dynamic team problems that admit static reductions. A certain structure on the cost function of a dynamic team and further assumptions on the corresponding RST problem are needed to prove the existence of a team-optimal solution. In the next subsection, we state the assumptions that we make on the dynamic team problem in order to establish existence.

5.1. Assumptions on dynamic team. In order to show the existence of optimal strategies in dynamic teams, we assume the following structure.

ASSUMPTION 5.2.
1. *The dynamic team problem satisfies Assumptions* 2.1, 2.2, *and* 5.1.
2. *The agents in the team do not share their observations with anyone. Any agent who acts more than once does not recall his/her past observation(s).*
3. *The cost function c of the dynamic team problem is in the structural form*

$$
c(\omega_0, \mathbf{y}_{1:T}^{1:N}, \mathbf{u}_{1:T}^{1:N}) = \sum_{t=1}^{T} \sum_{i=1}^{N} c_t^i(\mathbf{u}_t^i, \omega_0, \mathbf{u}_{1:t-1}^{1:N}, \mathbf{y}_{1:t-1}^{1:N}, \mathbf{y}_t^i) + \kappa(\omega_0, \mathbf{y}_{1:T}^{1:N}, \mathbf{u}_{1:T}^{1:N}),
$$

where c_t^i is a nonnegative and continuous function in the class $\mathrm{IC}(\Omega_0 \times \mathcal{U}_{1:t-1}^{1:N} \times \mathcal{Y}_{1:t-1}^{1:N} \times \mathcal{Y}_t^i, \mathcal{U}_t^i)$ for all $i \in [N]$ and $t \in [T]$, and κ is a nonnegative continuous function of its arguments.

4. For all $i \in [N]$ and $t \in [T]$, the continuous function $\varphi_t^i : \mathcal{Y}_t^i \times \Omega_0 \times \mathcal{U}_{1:t-1}^{1:N} \to \mathbb{R}^+$, as defined in Assumption 5.1, is strictly positive at all points in its domain.

Remark 5.1. The functional form of cost function (5.3) in Assumption 5.2 is common in many LQG teams with no observation sharing information structures. The reader is referred to section 7 for a few examples.

In the rest of this section, we consider dynamic teams satisfying Assumption 5.2. First, recall the following features of the corresponding RST:

1. If the behavioral strategy of agent i at time t is π_t^i, then the joint measure on $\mathcal{U}_t^i \times \mathcal{Y}_t^i$ in the corresponding RST problem is $\pi_t^i(d\mathbf{u}_t^i|\mathbf{y}_t^i)\nu_t^i(d\mathbf{y}_t^i)$.

2. Recall from Assumption 2.2 that there exists a set of behavioral strategies $\tilde{\pi}_{1:T}^{1:N}$ that achieves a finite cost $J(\tilde{\pi}_{1:T}^{1:N})$ in the dynamic team. Since RST and dynamic team problems are equivalent problems (see Theorem 5.3), RST also achieves the same cost with the behavioral strategies $\tilde{\pi}_{1:T}^{1:N}$.

Let $\{\mathcal{P}_t^i \subset \mathcal{R}_t^i\}_{i \in [N], t \in [T]}$ be the set of behavioral strategies of the agents such that there exists $\pi_{1:T}^{1:N}$ satisfying $\pi_t^i \in \mathcal{P}_t^i$ and $J(\pi_{1:T}^{1:N}) \leq J(\tilde{\pi}_{1:T}^{1:N})$. Define $\lambda_t^i(d\mathbf{u}_t^i, d\mathbf{y}_t^i) := \pi_t^i(d\mathbf{u}_t^i|\mathbf{y}_t^i)\nu_t^i(d\mathbf{y}_t^i)$ for $\pi_t^i \in \mathcal{P}_t^i$, and let $\mathcal{M}_t^i$ denote the set of all such λ_t^i.

If the team-optimal behavioral strategies exist, then the optimal behavioral strategy of agent (i, t) in the RST problem must lie in the set $\mathcal{M}_t^i$. In order to establish the existence of optimal strategies in the dynamic team, we show that $\mathcal{M}_t^i$ is a tight set of measures by using an approach similar to that in Lemma 4.4 and by using some auxiliary results proved in the next subsection.

5.2. Auxiliary results. Our first auxiliary result is as follows.

LEMMA 5.4.

1. For any $i \in [N]$ and $t \in [T]$ and any $\omega_0 \in \Omega_0$ and $\mathbf{u}_{1:t-1}^{1:N} \in \mathcal{U}_{1:t-1}^{1:N}$, we have

$$\int_{\mathcal{Y}_t^i \times \mathcal{U}_t^i} \varphi_t^i(\mathbf{y}_t^i; \omega_0, \mathbf{u}_{1:t-1}^{1:N})\lambda_t^i(d\mathbf{u}_t^i, d\mathbf{y}_t^i) = 1.$$

2. For any $i \in [N]$ and $t \in [T]$ and $\lambda_t^i \in \wp(\mathcal{Y}_t^i \times \mathcal{U}_t^i)$,

$$\int_{\Omega_0 \times \mathcal{Y}_{1:T}^{1:N} \times \mathcal{U}_{1:T}^{1:N}} c_t^i(\mathbf{u}_t^i, \omega_0, \mathbf{u}_{1:t-1}^{1:N}, \mathbf{y}_{1:t-1}^{1:N}, \mathbf{y}_t^i)\varphi(\omega_0, \mathbf{y}_{1:T}^{1:N}, \mathbf{u}_{1:T}^{1:N}) \, d\lambda_{1:T}^{1:N} \, \mathbb{P}\{d\omega_0\}$$

$$= \int_{\Omega_0 \times \mathcal{Y}_{1:t-1}^{1:N} \times \mathcal{U}_{1:t-1}^{1:N} \times \mathcal{Y}_t^i \times \mathcal{U}_t^i} \bar{c}_t^i(\mathbf{u}_t^i, \omega_0, \mathbf{u}_{1:t-1}^{1:N}, \mathbf{y}_{1:t-1}^{1:N}, \mathbf{y}_t^i) \, d\lambda_{1:t-1}^{1:N} \, d\lambda_t^i \, \mathbb{P}\{d\omega_0\},$$

where $\bar{c}_t^i(\mathbf{u}_t^i, \omega_0, \mathbf{u}_{1:t-1}^{1:N}, \mathbf{y}_{1:t-1}^{1:N}, \mathbf{y}_t^i) = c_t^i(\mathbf{u}_t^i, \omega_0, \mathbf{u}_{1:t-1}^{1:N}, \mathbf{y}_{1:t-1}^{1:N}, \mathbf{y}_t^i)$

$$(5.3) \qquad \times \varphi_t^i(\mathbf{y}_t^i; \omega_0, \mathbf{u}_{1:t-1}^{1:N}) \times \prod_{s=1}^{t-1} \prod_{j=1}^{N} \varphi_s^j(\mathbf{y}_s^j; \omega_0, \mathbf{u}_{1:s-1}^{1:N}).$$

Proof. The first statement holds for all $i \in [N]$ and $t \in [T]$ by the definition of φ_t^i in Assumption 5.1. The second statement is a consequence of the first statement. □

Recall that we introduced a class of functions $\mathrm{IC}(\cdot, \cdot)$ in Definition 4.2. In the next lemma, we show that if we multiply a function in this class with a lower-semicontinuous function that does not vanish in its domain, then the resulting function also belongs

to the same class. We use this result to show that the cost in (5.3) belongs to the class $\mathrm{IC}(\Omega_0 \times \mathcal{U}_{1:t-1}^{1:N} \times \mathcal{Y}_{1:t-1}^{1:N} \times \mathcal{Y}_t^i, \mathcal{U}_t^i)$.

LEMMA 5.5. *Let* $\phi_1 : \mathcal{A} \times \mathcal{B} \to \mathbb{R}$ *be a measurable function and let* $\phi_2 : \mathcal{A} \to \mathbb{R}^+$ *be a lower-semicontinuous function that is strictly positive everywhere in its domain. If* ϕ_1 *is in class* $\mathrm{IC}(\mathcal{A}, \mathcal{B})$, *then the product function* $\phi := \phi_1 \phi_2$ *is also in class* $\mathrm{IC}(\mathcal{A}, \mathcal{B})$.

Proof. Fix $m > 0$ and a compact set $\mathcal{K} \subset \mathcal{A}$. Define $M := \frac{m}{\min_{\mathbf{a} \in \mathcal{K}} \phi_2(\mathbf{a})}$. Then, by the property of class $\mathrm{IC}(\mathcal{A}, \mathcal{B})$ functions, there exists a compact set $\mathcal{L} \subset \mathcal{B}$, depending on $\mathcal{K}$ and M, such that $\inf_{\mathcal{K} \times \mathcal{L}^\complement} \phi_1(\mathbf{a}, \mathbf{b}) \geq M$. Now, due to the property of infimum, we get $\inf_{\mathcal{K} \times \mathcal{L}^\complement} \phi(\mathbf{a}, \mathbf{b}) \geq \inf_{\mathcal{K} \times \mathcal{L}^\complement} \phi_1(\mathbf{a}, \mathbf{b}) \min_{\mathcal{K}} \phi_2(\mathbf{a}) = m$, which completes the proof of the statement. ☐

As a result of the lemma above, we have the following fact.

LEMMA 5.6. *The function* $\bar{c}_t^i$, *as defined in* (5.3), *is a nonnegative and continuous function in the class* $\mathrm{IC}(\Omega_0 \times \mathcal{U}_{1:t-1}^{1:N} \times \mathcal{Y}_{1:t-1}^{1:N} \times \mathcal{Y}_t^i, \mathcal{U}_t^i)$.

Proof. By Assumption 5.2, $c_t^i(\mathbf{u}_t^i, \omega_0, \mathbf{u}_{1:t-1}^{1:N}, \mathbf{y}_{1:t-1}^{1:N}, \mathbf{y}_t^i)$ is in the class $\mathrm{IC}(\Omega_0 \times \mathcal{U}_{1:t-1}^{1:N} \times \mathcal{Y}_{1:t-1}^{1:N} \times \mathcal{Y}_t^i, \mathcal{U}_t^i)$. Define the functions ϕ_1 and ϕ_2 as

$$\phi_1(\mathbf{u}_t^i, \omega_0, \mathbf{u}_{1:t-1}^{1:N}, \mathbf{y}_{1:t-1}^{1:N}, \mathbf{y}_t^i) := c_t^i(\mathbf{u}_t^i, \omega_0, \mathbf{u}_{1:t-1}^{1:N}, \mathbf{y}_{1:t-1}^{1:N}, \mathbf{y}_t^i)$$

$$\phi_2(\omega_0, \mathbf{u}_{1:t-1}^{1:N}, \mathbf{y}_{1:t-1}^{1:N}, \mathbf{y}_t^i) := \varphi_t^i(\mathbf{y}_t^i; \omega_0, \mathbf{u}_{1:t-1}^{1:N}) \times \prod_{s=1}^{t-1} \prod_{j=1}^{N} \varphi_s^j(\mathbf{y}_s^j; \omega_0, \mathbf{u}_{1:s-1}^{1:N}),$$

and note that ϕ_2 is a continuous and strictly positive function in its domain (see part 4 of Assumption 5.2). Lemma 5.5 implies that the function $\bar{c}_t^i$ is nonnegative and continuous in the class $\mathrm{IC}(\Omega_0 \times \mathcal{U}_{1:t-1}^{1:N} \times \mathcal{Y}_{1:t-1}^{1:N} \times \mathcal{Y}_t^i, \mathcal{U}_t^i)$. ☐

We now have all the auxiliary results needed for showing existence. In the next subsection, we prove that any dynamic team as described above admits a team-optimal solution in deterministic strategies of the agents.

5.3. Existence of optimal strategies. Our first result here is as follows.

LEMMA 5.7. *The set of measures* $\mathcal{M}_t^i$ *is tight for all* $i \in [N]$ *and* $t \in [T]$.

Proof. See Appendix H. ☐

The lemma above implies that the search for a team-optimal solution in the RST problem can be restricted to the weak* precompact set $\mathcal{M}_{1:T}^{1:N}$. We now turn our attention to showing the existence of optimal strategies in the dynamic team.

THEOREM 5.8. *If a dynamic team problem satisfies Assumption* 5.2, *then it admits a team-optimal solution in deterministic strategies.*

Proof. Consider a sequence of behavioral strategies of the agents $\{(\pi_{1:T}^{1:N})_n\}_{n \in \mathbb{N}} \subset \mathcal{M}_{1:T}^{1:N}$ that satisfies $\lim_{n \to \infty} J((\pi_{1:T}^{1:N})_n) = \inf J(\pi_{1:T}^{1:N})$, where J is the expected cost functional of the dynamic team problem. Let $\{(\lambda_t^i)_n\}_{n \in \mathbb{N}} \subset \mathcal{M}_t^i$ be defined as

$$(\lambda_t^i)_n(d\mathbf{u}_t^i, d\mathbf{y}_t^i) = (\pi_t^i)_n(d\mathbf{u}_t^i | \mathbf{y}_t^i) \nu_t^i(d\mathbf{y}_t^i), \quad n \in \mathbb{N}.$$

Since $\{(\lambda_t^i)_n\}_{n \in \mathbb{N}}$ is a tight sequence of measures, we know that there exists a weak* convergent subsequence of measures. For every $i \in [N]$ and $t \in [T]$, let $\{(\lambda_t^i)_{n_k}\}_{k \in \mathbb{N}}$ be the weak* convergent subsequence of measures converging to $(\lambda_t^i)_0$. Define the behavioral strategy $(\pi_t^i)_0(d\mathbf{u}_t^i | \mathbf{y}_t^i) = (\lambda_t^i)_0(d\mathbf{u}_t^i | \mathbf{y}_t^i)$ for all $i \in [N]$ and $t \in [T]$. Since $c\varphi$ is a continuous function, from the result of Theorem 4.1, we conclude that

$$\liminf_{k \to \infty} \int c \, \varphi \, d(\lambda_{1:T}^{1:N})_{n_k} \mathbb{P}\{d\omega_0\} \geq \int c \, \varphi \, d(\lambda_{1:T}^{1:N})_0 \mathbb{P}\{d\omega_0\},$$

or, equivalently, $\liminf_{k\to\infty} J((\pi_{1:T}^{1:N})_{n_k}) \geq J((\pi_{1:T}^{1:N})_0)$. Thus, optimal behavioral strategies exist in the RST, and the optimal behavioral strategy of agent (i,t) is the conditional measures $(\pi_t^i)_0(d\mathbf{u}^i|\mathbf{y}^i)$. Since RST is equivalent to the dynamic team, this is also the optimal behavioral strategy of agent i at time t in the dynamic team.

Moreover, by Blackwell's irrelevant information theorem [37, p. 457], there exists a set of deterministic strategies which achieves the same cost as that achieved using optimal behavioral strategies of the agents, which establishes the result.   □

We have the following corollary to the theorem above for dynamic teams in which the agents have compact action spaces. We do not require parts 3 and 4 of Assumption 5.2 to show the existence of optimal strategies for this problem.

COROLLARY 5.9. *Let us consider a dynamic team problem in which $\mathcal{U}_t^i$ is compact for all $i \in [N]$ and $t \in [T]$. If parts 1 and 2 in Assumption 5.2 hold, then the dynamic team problem admits an optimal solution in deterministic strategies of the agents.*

Proof. As a result of Assumption 5.1, the dynamic team problem is equivalent to the RST defined in Definition 5.2. Note that due to the assumption, the cost function of the RST is continuous. Applying the result of Corollary 4.7 to the RST, we conclude that it admits a team-optimal solution in deterministic strategies. The optimal strategy of agent (i,t) in the RST is also the optimal strategy of agent i at time t in the dynamic team due to the equivalence of the two team problems.   □

We now revisit Witsenhausen's counterexample.

Revisiting Witsenhausen's counterexample. Recall that we were unable to prove the existence of a solution to the corresponding RST problem of Witsenhausen's counterexample using the results we obtained for static team problems in sections 3 and 4. We now outline the essential steps of the proof above adapted to the RST of Witsenhausen's counterexample.

Let $\nu_1 \in \wp(\mathcal{Y}_1)$ and $\nu_2 \in \wp(\mathcal{Y}_2)$ be probability measures that admit mean-zero unit variance Gaussian density functions. First, note that if both agents apply zero control, then the expected cost is $\mathbb{E}[Y_1^2] = 1$, which is finite. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be the behavioral strategy spaces of the first and the second controller, respectively. Let $\mathcal{P}_1 \subset \mathcal{R}_1$ and $\mathcal{P}_2 \subset \mathcal{R}_2$ be the sets of behavioral strategies of the controllers such that there exist $\pi_1 \in \mathcal{P}_1$ and $\pi_2 \in \mathcal{P}_2$ which yield $J(\pi_1, \pi_2) \leq 1$. Now, the following four steps lead to the existence of a team-optimal solution to this problem with nonclassical information.

1. Recall the expression of the cost function of the corresponding RST from (5.2). For any $\pi_2 \in \mathcal{R}_2$, we have

$$\int_{\mathcal{Y}_2 \times \mathcal{U}_2} c_S \, \pi_2(du_2|y_2)\nu_2(dy_2)$$
$$\geq \int_{\mathcal{Y}_2 \times \mathcal{U}_2} (u_1 - y_1)^2 \exp\left(\frac{-u_1^2 + 2y_2 u_1}{2}\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y_2^2}{2}\right) \pi_2(du_2|y_2)dy_2,$$
$$= (u_1 - y_1)^2 \int_{\mathcal{Y}_2 \times \mathcal{U}_2} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-(y_2 - u_1)^2}{2}\right) \pi_2(du_2|y_2)dy_2$$
$$= (u_1 - y_1)^2,$$

where the first inequality follows from dropping the quadratic term $(u_2 - u_1)^2$ from the expression of c_S, the second equality is immediate, and the third equality follows from the fact that $\pi_2(du_2|y_2) \frac{1}{\sqrt{2\pi}} \exp(\frac{-(y_2 - u_1)^2}{2})dy_2$ is a probability measure over $\mathcal{U}_2 \times \mathcal{Y}_2$. This is also a consequence of Lemma 5.4.

2. The function $(u_1 - y_1)^2$ is in class $\mathrm{IC}(\mathcal{Y}_1, \mathcal{U}_1)$ and $(u_2 - u_1)^2 \exp(\frac{-u_1^2 + 2y_2 u_1}{2})$ is in class $\mathrm{IC}(\mathcal{U}_1 \times \mathcal{Y}_1 \times \mathcal{Y}_2, \mathcal{U}_2)$ by Lemma 5.6.

3. Recall that there exist $\pi_1 \in \mathcal{P}_1$ and $\pi_2 \in \mathcal{P}_2$ such that $J(\pi_1, \pi_2) \leq 1$. Thus, the set of measures $\mathcal{M}_i$, defined by

$$\mathcal{M}_i := \left\{ \lambda_i \in \wp(\mathcal{Y}_i \times \mathcal{U}_i) : \lambda_i(du_i, dy_i) = \pi_i(du_i|y_i)\nu_i(dy_i), \ \pi_i \in \mathcal{P}_i \right\},$$

is tight for $i \in \{1, 2\}$ by Lemma 5.7. The proof essentially uses points 1 and 2 above, coupled with Lemma 4.3 in a sequential fashion. Using point 1, we conclude that

$$\int_{\mathcal{U}_1 \times \mathcal{Y}_1} (u_1 - y_1)^2 \lambda_1(du_1, dy_1) \leq \mathbb{E}\left[Y_1^2\right] = 1$$

for all $\lambda_1 \in \mathcal{M}_1$ (or equivalently, $\pi_1 \in \mathcal{P}_1$). Then, using point 2 and Lemma 4.3, we conclude that the set of measures $\mathcal{M}_1$ is tight. Now, notice that

$$\int_{\mathcal{Y}_{1:2} \times \mathcal{U}_{1:2}} (u_2 - u_1)^2 \exp\left(\frac{-u_1^2 + 2y_2 u_1}{2}\right) d\lambda_1 \, d\lambda_2 \leq \mathbb{E}\left[Y_1^2\right] = 1.$$

Since $\mathcal{M}_1$ is tight and $(u_2 - u_1)^2 \exp(\frac{-u_1^2 + 2y_2 u_1}{2})$ is in class $\mathrm{IC}(\mathcal{U}_1 \times \mathcal{Y}_1 \times \mathcal{Y}_2, \mathcal{U}_2)$ (see point 2 above), we conclude that $\mathcal{M}_2$ is tight by Lemma 4.3.

4. Finally, using the same arguments as those in the proof of Theorem 4.5, one can conclude that there exist optimal strategies of the agents.

Notice that the above proof of existence of a solution to Witsenhausen's counterexample is completely different from the proofs given in [33] and [36].

This concludes the discussion in this section. In the next section, we show the existence of optimal solution in LQG team problems with "no observation sharing" information structures using the results of this section.

6. LQG teams. We now consider a class of dynamic teams in which the state, action, and observation spaces are Euclidean spaces, the state transition and observation functions are linear, and the primitive random variables are mutually independent Gaussian random variables. In particular, we assume that the observation equation for agent (i, t) is given by

$$(6.1) \qquad \qquad \mathbf{y}_t^i = h_t^i(\omega_0, \mathbf{u}_{1:t-1}^{1:N}) + \omega_t^i,$$

where h_t^i is a linear function of its arguments and ω_t^i is a zero-mean Gaussian random vector with positive-definite covariance.

We assume that the cost function of the dynamic team problem is quadratic in the actions of the agents and is of the following form:

(6.2)

$$c(\omega_0, \mathbf{y}_{1:T}^{1:N}, \mathbf{u}_{1:T}^{1:N}) = \sum_{t=1}^{T} \sum_{i=1}^{N} \|\mathbf{u}_t^i - p_t^i(\omega_0, \mathbf{u}_{1:t-1}^{1:N}, \mathbf{y}_{1:t-1}^{1:N}, \mathbf{y}_t^i)\|_{R_t^i}^2 + \kappa(\omega_0, \mathbf{y}_{1:T}^{1:N}, \mathbf{u}_{1:T}^{1:N}),$$

where $\{R_t^i\}_{i \in [N], t \in [T]}$ is a sequence of positive-definite matrices of appropriate dimensions, $\{p_t^i : \Omega_0 \times \mathcal{U}_{1:t-1}^{1:N} \times \mathcal{Y}_{1:t-1}^{1:N} \times \mathcal{Y}_t^i \to \mathcal{U}_t^i\}_{i \in [N], t \in [T]}$ is a sequence of continuous

functions,[4] and κ is a nonnegative continuous function. We henceforth refer to teams satisfying the above assumptions and having a cost function of the form (6.3) as LQG teams with no observation sharing, and we address existence of team-optimal solutions below.

THEOREM 6.1. *Consider a dynamic LQG team as formulated above, where the agents do not share their observations and the observation of each agent as given by* (6.1) *is corrupted by additive Gaussian noise. If the cost is given by* (6.3)*, then the dynamic LQG team admits a team-optimal solution in deterministic strategies.*

Proof. In order to establish the result, we need to verify that all parts of Assumption 5.2 are satisfied by the LQG team problem.

The linearity of state transition and observation equations implies that Assumption 2.1 is satisfied and that $\{h_t^i\}_{i\in[N],t\in[T]}$, as defined in (6.1), are continuous functions. If we apply zero control action, then the expected cost is finite because the cost is quadratic in the primitive random variables and their distributions are Gaussian. Thus, Assumption 2.2 is satisfied. Since the observation noises are additive and Gaussian, Assumption 5.1 is satisfied. Furthermore, due to the Gaussian nature of observation noise, we also conclude that φ_t^i is strictly positive at all points in its domain for all $i \in [N]$ and $t \in [T]$ (see the proof of Lemma 5.1).

The cost function c is continuous. Since p_t^i is continuous and R_t^i is positive definite, the function $\|\mathbf{u}_t^i - p_t^i(\omega_0, \mathbf{u}_{1:t-1}^{1:N}, \mathbf{y}_{1:t-1}^{1:N}, \mathbf{y}_t^i)\|_{R_t^i}^2$ lies in the class $\mathrm{IC}(\Omega_0 \times \mathcal{U}_{1:t-1}^{1:N} \times \mathcal{Y}_{1:t-1}^{1:N} \times \mathcal{Y}_t^i, \mathcal{U}_t^i)$ for all $i \in [N]$ and $t \in [T]$. The statement is then simply a consequence of Theorem 5.8. $\quad\square$

We have thus identified conditions under which an LQG team problem admits an optimal solution. In the next section, we consider a number of well-studied LQG team problems from the literature and establish the existence of team-optimal strategies. Existence of optimal strategies in some of the team problems formulated in the next section is established here for the first time.

7. Examples. In this section, we present some examples of LQG teams with the "no observation sharing" information structure. In all the examples, Theorem 6.1 leads to the conclusion that team-optimal strategies exist. Except for scalar Witsenhausen's counterexample and the Gaussian test channel, existence of optimal strategies was not known for any of the LQG teams considered in this section.

7.1. One-agent finite horizon (static output feedback) LQG problem. Consider a linear system in which all primitive random variables are Gaussian and mutually independent of each other. The agent has a stagewise additive quadratic cost function. The information available to the controller at time t is $\mathbf{Y}_t$, where $\mathbf{Y}_t = H_t \mathbf{X}_t + \mathbf{W}_t$ for some matrix H_t of appropriate dimensions, that is, we have a static output feedback problem. The total cost to the controller is

$$c(\mathbf{x}_{1:T+1}, \mathbf{u}_{1:T}) = \sum_{t=1}^{T} \left(\mathbf{x}_{t+1}^{\mathsf{T}} Q \mathbf{x}_{t+1} + \mathbf{u}_t^{\mathsf{T}} R \mathbf{u}_t \right), \qquad Q \geq 0, \quad R > 0.$$

Since this is an LQG problem with no sharing of observation, it satisfies both hypotheses of Theorem 6.1. Using Theorem 6.1, we conclude that an optimal static output feedback controller exists. This solution, however, need not be linear [3].

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[4]In most cases of interest, $\{p_t^i\}_{i\in[N],t\in[T]}$ are linear maps, which is the reason why we have called this class of teams LQG, realizing that, in general, with p_t^i's nonlinear, c is not going to be quadratic in the u_t^i's.

7.2. The Gaussian test channel. The Gaussian test channel consists of an encoder and a decoder. The source observes a zero-mean Gaussian random variable X_1 with variance σ_1^2, which is encoded by the encoder (agent 1), and the encoded symbol U_1 is sent across a noisy channel to a decoder. The additive noise W_2 on the channel is assumed to be a zero mean Gaussian random variable with variance σ_w^2. The decoder (agent 2) observes the corrupted message Y_2 and estimates the realization of the random variable X_1 available at the source. The decoder's estimate is denoted by U_2.

The information structure of the encoder is $I_1^1 = \{X_1\}$ and of the decoder is $I_2^2 = \{Y_2\}$. Thus, this is an example of a team with asymmetric information. Note that the observation of agent 2 satisfies the observation equation (6.1). Lemma 5.1 implies that this dynamic team problem admits a static reduction. The cost function of the team of encoder and decoder is[5] $c(x_1, u_1, u_2) = \lambda u_1^2 + (u_2 - x_1)^2$, where $\lambda > 0$.

One can check that the cost function of the team is of the form in (6.3). It is well known that the optimal encoding and decoding strategies are linear in their arguments, despite the fact that the information structure is nonnested. The only known proof of this result (and therefore of the existence of a solution to this team problem) is an indirect one that uses information theoretic concepts; see, for example, [5]. We now have here another proof of the existence of team-optimal strategies for the Gaussian test channel as a consequence of Theorem 6.1. The existence result also holds for the more general two-agent LQG problem introduced in [5], which subsumes the Gaussian test channel and Witsenhausen's counterexample as special cases. For such extensions, see also [4].
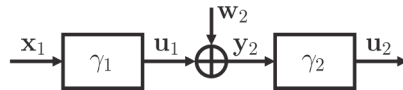

FIG. 1. *A figure depicting the unified setup of the Gaussian test channel and Witsenhausen's counterexample from* [5].

7.3. Multidimensional Gaussian test channel and Witsenhausen's counterexample. Consider the setup depicted in Figure 1, with a difference that all random vectors take values in finite dimensional Euclidean spaces of appropriate dimensions. Furthermore, we assume that $\mathbf{W}_2$ has a strictly positive-definite covariance, and the entries in $\mathbf{W}_2$ can be correlated. Consider the cost function of the team as

$$c(\mathbf{x}_1, \mathbf{u}_1, \mathbf{u}_2) = \lambda \|\mathbf{u}_1\|^2 + \|\mathbf{u}_2 - H\mathbf{x}_1\|^2, \qquad \lambda > 0,$$

where H is a matrix of appropriate dimensions.

It has been shown that under some specific assumptions on the covariance matrix of the noise variable $\mathbf{W}_2$, optimal encoding and decoding schemes exist in the multidimensional Gaussian test channel, again using information theoretic tools; see [37, section 11.2.3] and references therein for a review of such results. In particular, if certain "matching conditions" hold, that is, if the rate distortion achieving transition kernel is matched with the channel capacity achieving source distribution (see

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[5] For the Gaussian test channel, this corresponds to the "soft-constrained" version; the standard version has a second moment (hard) constraint on u_1. One can show, however, that existence of an optimal solution to one implies existence to the other and vice versa.

[15] and Remark 11.2.1 in [37] in the context of Gaussian systems), then optimal encoding-decoding strategies will exist.

As in the scalar case, the multidimensional Gaussian test channel admits static reduction, and the cost function has the same form as in (6.3). Theorem 6.1 implies that optimal encoding-decoding strategies exist even if $\mathbf{X}_1$, $\mathbf{U}_1$, and $\mathbf{U}_2$ take values in different Euclidean spaces. Thus, a large class of multidimensional Gaussian test channel problems admits optimal solutions. Similar arguments prove that team-optimal solution exists for vector Witsenhausen's counterexample, which includes the one studied in [16].

7.4. A Gaussian relay channel. Consider now the Gaussian relay channel, depicted in Figure 2, that was studied in [24] and [39]. It comprises an encoder, a certain number of relays, and a decoder. This problem was formulated in [24] where the authors have shown that nonlinear strategies outperform linear strategies when there are two or more relays. Zaidi et al. studied this problem in [39], and they showed that, in fact, even with one relay, quantization-based strategies outperform linear strategies of the agents. Thus, linear strategies of encoder and decoder are optimal only in the case of the Gaussian test channel discussed earlier, but not in the case of the Gaussian relay channel.
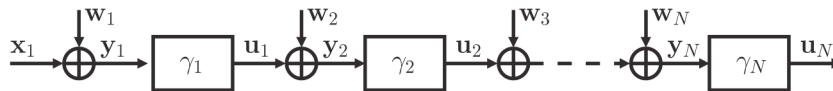

FIG. 2. *A figure depicting the arrangement of encoder, relays, and decoder in the Gaussian relay channel. Agent 1 is the encoder, and agent N is the decoder, while the agents from 2 to N − 1 are relays in the figure.*

For a concrete formulation of the problem, assume that there are $N - 2$ relays and all state, action, noise, and observation spaces are the real line. The encoder observes a noise corrupted version of a zero-mean Gaussian random variable, X_1, with variance σ_1^2. With the observation noise of the encoder denoted by W_1, the observation of the encoder is $Y_1 := X_1 + W_1$. The encoder's information is $I^1 = \{Y_1\}$ and the action of the encoder is U_1. For $i \in \{2, \ldots, N - 1\}$, the ith relay receives a noise corrupted version of the transmitted signal, denoted by $Y_i := U_{i-1} + W_i$, and $I^i = \{Y_i\}$. The ith relay outputs U_i. Finally, the decoder receives Y_N and outputs U_N, which is an estimate of the realization of the random variable X_1. The noise variables $W_i, i \in [N]$, are assumed to be pairwise independent, mean-zero Gaussian random variables with some specified variances, and independent of the random variable X_1. Since the observations of the agents $2, \ldots, N-1$ satisfy the observation equation (6.1), we conclude that the dynamic team problem admits a static reduction using Lemma 5.1. The cost function of the team is $c(x_1, u_{1:N}) = (u_N - x_1)^2 + \sum_{i=1}^{N-1} \lambda_i u_i^2$, where $\lambda_i > 0$ for all $i \in [N]$. It is known for this problem that for any number of relays, nonlinear strategies outperform linear ones. However, it was not known until now whether there existed optimal strategies for the agents.

Since this is an LQG team with no observation sharing information structure and the cost function of the team is of the form in (6.3), we conclude that optimal encoding, decoding, and relay strategies exist by the result of Theorem 6.1.

8. Conclusion. In this paper, we have identified a set of sufficient conditions on a stochastic team problem with "no observation sharing" information structure that

guarantee existence of an optimal solution. In particular, in a static team problem, if the cost function is continuous and has a certain structure, and the observation channels satisfy certain technical conditions, then there exists an optimal solution.

We used Witsenhausen's static reduction technique to obtain a similar result for a class of dynamic teams with no sharing of observations among the agents. Furthermore, we proved that LQG teams with no sharing of observations admit team-optimal solutions under some technical conditions on the cost functions. As a consequence of one of the main results of the paper, we also showed that several dynamic LQG team problems from the literature admit team-optimal solutions. The approach developed in the paper and the specific results obtained settle for good a number of open questions on the existence of optimal strategies in dynamic teams. Furthermore, the results of this paper can be applied to optimal real-time coding/decoding problems to prove the existence of optimal strategies, where it was not been known earlier whether optimal policies exist, and the few special cases where such results exist have relied on strict information theoretic source-channel matching conditions.

As for future work, one goal is to obtain approximately optimal policies and numerical techniques to make such optimization problems tractable. Another goal is to obtain explicit analytical solutions for some classes of dynamic teams. Obtaining conditions under which a team-optimal solution exists in a team with "observation sharing" information structure is also an important area for further research. Finally, informational aspects of noncooperative games is a further relevant area of study.

Appendix A. Proof of Lemma 3.2. Since g is uniformly continuous, we can assume that for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $\mathbf{a}_0 \in \mathcal{A}$, $\mathbf{c}_0 \in \mathcal{C}$ and $\mathbf{a} \in \mathcal{A}$, $\mathbf{c} \in \mathcal{C}$ satisfying $d_{\mathcal{A}}(\mathbf{a}, \mathbf{a}_0) < \delta$, $d_{\mathcal{C}}(\mathbf{c}, \mathbf{c}_0) < \delta$, we have $\sup_{\mathbf{b} \in \mathcal{B}} |g(\mathbf{a}, \mathbf{b}, \mathbf{c}) - g(\mathbf{a}_0, \mathbf{b}, \mathbf{c}_0)| < 2\epsilon$. Let $M := \sup_{\mathbf{a} \in \mathcal{A}} \int h_{(\eta, \nu)} d\nu < \infty$. Consider any probability measure $\mu \in \wp(\mathcal{Y} \times \mathcal{B})$. Then, we get

$$|f_\mu(\mathbf{a}, \mathbf{c}) - f_\mu(\mathbf{a}_0, \mathbf{c}_0)|$$
$$= \left| \int_{\mathcal{Y} \times \mathcal{B}} g(\mathbf{a}, \mathbf{b}, \mathbf{c})\mu(d\mathbf{b}|\mathbf{y})\eta(\mathbf{a}, \mathbf{y})\nu(d\mathbf{y}) - \int_{\mathcal{Y} \times \mathcal{B}} g(\mathbf{a}_0, \mathbf{b}, \mathbf{c}_0)\mu(d\mathbf{b}|\mathbf{y})\eta(\mathbf{a}_0, \mathbf{y})\nu(d\mathbf{y}) \right|,$$
$$\leq \int_{\mathcal{Y} \times \mathcal{B}} \left| g(\mathbf{a}, \mathbf{b}, \mathbf{c})\eta(\mathbf{a}, \mathbf{y}) - g(\mathbf{a}_0, \mathbf{b}, \mathbf{c}_0)\eta(\mathbf{a}_0, \mathbf{y}) \right| \mu(d\mathbf{b}|\mathbf{y})\nu(d\mathbf{y}),$$
$$\leq \int_{\mathcal{Y} \times \mathcal{B}} \left(g(\mathbf{a}_0, \mathbf{b}, \mathbf{c}_0) \left| \eta(\mathbf{a}, \mathbf{y}) - \eta(\mathbf{a}_0, \mathbf{y}) \right| + \left| g(\mathbf{a}, \mathbf{b}, \mathbf{c}) - g(\mathbf{a}_0, \mathbf{b}, \mathbf{c}_0) \right| \eta(\mathbf{a}, \mathbf{y}) \right)$$
$$\mu(d\mathbf{b}|\mathbf{y})\nu(d\mathbf{y}),$$
$$\leq \|g\|_\infty M\epsilon + \int_{\mathcal{Y} \times \mathcal{B}} \left| g(\mathbf{a}, \mathbf{b}, \mathbf{c}) - g(\mathbf{a}_0, \mathbf{b}, \mathbf{c}_0) \right| \mu(d\mathbf{b}|\mathbf{y})\eta(\mathbf{a}_0, \mathbf{y})\nu(d\mathbf{y}),$$
$$< (\|g\|_\infty M + 2)\epsilon.$$

Also notice that $\{f_\mu(\cdot, \cdot)\}_{\mu \in \wp(\mathcal{Y} \times \mathcal{B})}$ is bounded by $\|g\|_\infty$, which completes the proof. $\square$

Appendix B. Proof of Lemma 3.4. We mimic the steps of the proof of Lemma 3.2 to prove this statement. First, notice that $\{f_{\mu^{1:N}}(\cdot, \cdot)\}_{\mu^i \in \wp(\mathcal{Y}^i \times \mathcal{B}^i)}$ is uniformly bounded by $\|g\|_\infty$. Now, we prove that this family of functions is equicontinuous.

Let us define η, M, and ν as

$$\eta(\mathbf{a}, \mathbf{y}^{1:N}) := \prod_{i=1}^N \eta^i(\mathbf{a}, \mathbf{y}^i), \quad M = \max_{i \in [N]} \sup_{\mathbf{a} \in \mathcal{A}} \int_{\mathcal{Y}^i} h^i d\nu^i, \quad \nu(d\mathbf{y}^{1:N}) = \prod_{i=1}^N \nu^i(d\mathbf{y}^i).$$

First, note that for any $\mathbf{a} \in \mathcal{A}$ and $i \in [N]$,

$$(\text{B.1}) \qquad \int_{\mathcal{Y}^i} \eta^i(\mathbf{a}, \mathbf{y}^i) \nu^i(d\mathbf{y}^i) = \int_{\mathcal{Y}^i} \mathbb{P}\{d\mathbf{y}^i | \mathbf{a}\} = 1.$$

Let $\epsilon > 0$. Let $\delta^i > 0$ be such that for any $d_{\mathcal{A}}(\mathbf{a}, \mathbf{a}_0) < \delta^i$, we have $|\eta^i(\mathbf{a}, \mathbf{y}^i) - \eta^i(\mathbf{a}_0, \mathbf{y}^i)| < \epsilon\, h^i(\mathbf{a}_0, \mathbf{y}^i)$. Pick $\delta = \min_{i \in [N]} \delta^i$. Fix $\mathbf{a}_0 \in \mathcal{A}$. Now, for $\mathbf{a} \in \mathcal{A}$ such that $d_{\mathcal{A}}(\mathbf{a}, \mathbf{a}_0) < \delta$, note the following:

$$\left| \eta(\mathbf{a}, \mathbf{y}^{1:N}) - \eta(\mathbf{a}_0, \mathbf{y}^{1:N}) \right|$$

$$\leq \sum_{j=1}^{N} \left(\prod_{i=1}^{j-1} \eta^i(\mathbf{a}_0, \mathbf{y}^i) \, \left| (\eta^j(\mathbf{a}, \mathbf{y}^j) - \eta^j(\mathbf{a}_0, \mathbf{y}^j)) \right| \prod_{i=j+1}^{N} \eta^i(\mathbf{a}, \mathbf{y}^i) \right)$$

$$\leq \epsilon \left(\sum_{j=1}^{N} \left(\prod_{i=1}^{j-1} \eta^i(\mathbf{a}_0, \mathbf{y}^i) \prod_{i=j+1}^{N} \eta^i(\mathbf{a}, \mathbf{y}^i) \right) h^j(\mathbf{a}_0, \mathbf{y}^j) \right),$$

where terms with $\prod_{i=1}^0$ and $\prod_{i=N+1}^N$ are replaced by 1. Use the expression to get

$$\left| \int_{\mathcal{B}^{1:N} \times \mathcal{Y}^{1:N}} g(\mathbf{a}, \mathbf{b}^{1:N}, \mathbf{c}) \left(\prod_{i=1}^{N} \mu^i(d\mathbf{b}^i | \mathbf{y}^i) \right) (\eta(\mathbf{a}, \mathbf{y}^{1:N}) - \eta(\mathbf{a}_0, \mathbf{y}^{1:N}))\, \nu(d\mathbf{y}^{1:N}) \right|$$

$$< \epsilon \|g\|_\infty \sum_{j=1}^{N} \int_{\mathcal{Y}^{1:N}} \left(\prod_{i=1}^{j-1} (\eta^i(\mathbf{a}_0, \mathbf{y}^i) d\nu^i) \prod_{i=j+1}^{N} (\eta^i(\mathbf{a}, \mathbf{y}^i) d\nu^i) \right) h^j(\mathbf{a}_0, \mathbf{y}^j) d\nu^j$$

$$(\text{B.2}) \qquad \leq \epsilon \|g\|_\infty N M,$$

where we also used (B.1). Since g is uniformly continuous, for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $\mathbf{a}_0 \in \mathcal{A}$, $\mathbf{c}_0 \in \mathcal{C}$, and $\mathbf{a} \in \mathcal{A}$, $\mathbf{c} \in \mathcal{C}$ satisfying $d_{\mathcal{A}}(\mathbf{a}, \mathbf{a}_0) < \delta$, $d_{\mathcal{C}}(\mathbf{c}, \mathbf{c}_0) < \delta$, we have $\sup_{\mathbf{b}^{1:N} \in \mathcal{B}^{1:N}} |g(\mathbf{a}, \mathbf{b}^{1:N}, \mathbf{c}) - g(\mathbf{a}_0, \mathbf{b}^{1:N}, \mathbf{c}_0)| < 2\epsilon$. Using this inequality and (B.2), one can show that

$$|f_{\mu^{1:N}}(\mathbf{a}, \mathbf{c}) - f_{\mu^{1:N}}(\mathbf{a}_0, \mathbf{c}_0)| < (\|g\|_\infty N M + 2)\epsilon,$$

which establishes the result. $\quad\square$

Appendix C. Proof of Lemma 3.5. In order to prove the lemma, we first need the following result.

LEMMA C.1. *Let $\mathcal{A}$ be a Polish space. Let $\{h_n : \mathcal{A} \to \mathbb{R}\}_{n \in \mathbb{N}}$ be a convergent sequence of continuous and uniformly bounded functions, and let $h_0 : \mathcal{A} \to \mathbb{R}$ be a continuous function such that for any compact subset $\mathcal{A} \subset \mathcal{A}$, $\sup_{\mathbf{a} \in \mathcal{A}} |h_n(\mathbf{a}) - h_0(\mathbf{a})| \to 0$ as $n \to \infty$. Let $\{\mu_n\}_{n \in \mathbb{N} \cup \{0\}} \subset \wp(\mathcal{A})$ be a weak* convergent sequence of measures such that $\mu_n \xrightarrow{w^*} \mu_0$ as $n \to \infty$. Then,*

$$\lim_{n \to \infty} \int_{\mathcal{A}} h_n d\mu_n = \int_{\mathcal{A}} h_0 d\mu_0 \quad \text{and} \quad \lim_{n \to \infty} \left| \int_{\mathcal{A}} h_n d\mu_n - \int_{\mathcal{A}} h_n d\mu_0 \right| = 0.$$

Proof. Since $\{\mu_n\}_{n \in \mathbb{N}}$ is a weak* convergent sequence, it is tight, which further implies that for any $\epsilon > 0$, there exists a compact set $\mathcal{A}_\epsilon \subset \mathcal{A}$ such that $\mu_n(\mathcal{A}_\epsilon^c) < \epsilon$ for every $n \in \mathbb{N} \cup \{0\}$. Fix $\epsilon > 0$. Since h_n converges uniformly to h_0 over the compact set $\mathcal{A}_\epsilon$, there exists $N_{\epsilon,1} \in \mathbb{N}$ such that

$$|h_n(\mathbf{a}) - h_0(\mathbf{a})| < \epsilon \quad \text{for all } \mathbf{a} \in \mathcal{A}_\epsilon \text{ and } n \geq N_{\epsilon,1}.$$

Let M be the uniform bound on the sequence of functions $\{h_n\}_{n\in\mathbb{N}}$. Then, $\|h_0\|_\infty \leq M$. Thus, for any $n \geq N_{\epsilon,1}$, we get

$$\left|\int_{\mathcal{A}} h_n d\mu_n - \int_{\mathcal{A}} h_0 d\mu_n\right| \leq \int_{\mathcal{A}_\epsilon} |h_n(\mathbf{a}) - h_0(\mathbf{a})| d\mu_n(\mathbf{a}) + \int_{\mathcal{A}_\epsilon^{\complement}} |h_n(\mathbf{a}) - h_0(\mathbf{a})| d\mu_n(\mathbf{a})$$

$$\text{(C.1)} \qquad\qquad\qquad < (1 + 2M)\epsilon.$$

By the definition of weak* convergence of measures, there exists $N_{\epsilon,2} \in \mathbb{N}$ such that

$$\left|\int_{\mathcal{A}} h_0 d\mu_n - \int_{\mathcal{A}} h_0 d\mu_0\right| < \epsilon \quad \text{for all } n \geq N_{\epsilon,2}.$$

Take $N_\epsilon = \max\{N_{\epsilon,1}, N_{\epsilon,2}\}$. Now, for $n > N_\epsilon$, we have

$$\left|\int_{\mathcal{A}} h_n d\mu_n - \int_{\mathcal{A}} h_0 d\mu_0\right| \leq \left|\int_{\mathcal{A}} h_n d\mu_n - \int_{\mathcal{A}} h_0 d\mu_n\right| + \left|\int_{\mathcal{A}} h_0 d\mu_n - \int_{\mathcal{A}} h_0 d\mu_0\right|$$

$$\text{(C.2)} \qquad\qquad\qquad < 2(1 + M)\epsilon.$$

Now, consider the following inequalities for $n \geq N_\epsilon$:

$$\left|\int_{\mathcal{A}} h_n d\mu_n - \int_{\mathcal{A}} h_n d\mu_0\right| \leq \left|\int_{\mathcal{A}} h_n d\mu_n - \int_{\mathcal{A}} h_0 d\mu_0\right| + \left|\int_{\mathcal{A}} h_0 d\mu_0 - \int_{\mathcal{A}} h_n d\mu_0\right|$$

$$< 2(1 + M)\epsilon + (1 + 2M)\epsilon = (3 + 4M)\epsilon,$$

where the first inequality is just the triangle inequality on the real line, whereas the second inequality follows from (C.1) and (C.2). $\qquad\square$

Now, we turn our attention to the proof of Lemma 3.5. By Lemma 3.4, we know that $\{f_n\}$ is a sequence of uniformly equicontinuous and uniformly bounded functions. In this step, we show that there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ and $f_0 \in C_b(\mathcal{A} \times \mathcal{C})$ such that f_{n_k} converges to f_0 uniformly over any compact set in $\mathcal{A} \times \mathcal{C}$.

Since $\mathcal{A} \times \mathcal{C}$ is σ-compact, there exists a countable collection of compact sets $\mathcal{K}_n \subset \mathcal{A} \times \mathcal{C}$ such that $\mathcal{A} \times \mathcal{C} = \cup_{n\in\mathbb{N}}\mathcal{K}_n$. Let $\mathcal{L}_m = \cup_{k=1}^m \mathcal{K}_k$. By the Arzela–Ascoli theorem [1], for every $m \in \mathbb{N}$ there exists a convergent subsequence $\{f_{n_k^m}\}_{k\in\mathbb{N}}$ and a continuous function $f_0^m : \mathcal{L}_m \to \mathbb{R}$ such that $\sup_{(\mathbf{a},\mathbf{c})\in\mathcal{L}_m} |f_{n_k^m}(\mathbf{a},\mathbf{c}) - f_0^m(\mathbf{a},\mathbf{c})| \to 0$ as $k \to \infty$. We can take $\{n_k^{m+1}\}_{k\in\mathbb{N}}$ to be a subsequence of $\{n_k^m\}_{k\in\mathbb{N}}$ for every $m \in \mathbb{N}$. Now, since $\mathcal{L}_m \subset \mathcal{L}_{m+1}$, we conclude that f_0^m agrees with f_0^{m+1} on set $\mathcal{L}_m$ for every $m \in \mathbb{N}$. Using Cantor's diagonalization argument, we get a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ and a continuous function f_0 such that $f_{n_k} \to f_0$, where the convergence is uniform over any compact set in $\mathcal{A} \times \mathcal{C}$. Furthermore, since f_n is uniformly bounded, f_0 is also bounded. We now use Lemma C.1 to establish the statement of Lemma 3.5.

Appendix D. Proof of Lemma 3.6. First, note that since pullback of a measure is a continuous operation [2], $\mathsf{Pr}_\#^{\mathcal{B}\times\mathcal{C}}\mu_n \to \mathsf{Pr}_\#^{\mathcal{B}\times\mathcal{C}}\mu_0$. Pick any $g \in U_b(\mathcal{A}\times\mathcal{B}\times\mathcal{C})$ and note that $(\mathbf{b},\mathbf{c}) \mapsto \int_{\mathcal{A}} g(\mathbf{a},\mathbf{b},\mathbf{c})\rho(\mathbf{a},\mathbf{b})\nu(d\mathbf{a})$ is a bounded uniformly continuous function from Corollary 3.3. This gives

$$\int g d\mu_0 = \lim_{n\to\infty}\int g d\mu_n = \lim_{n\to\infty}\int_{\mathcal{B}\times\mathcal{C}}\left(\int_{\mathcal{A}} g(\mathbf{a},\mathbf{b},\mathbf{c})\rho(\mathbf{a},\mathbf{b})\nu(d\mathbf{a})\right)\mathsf{Pr}_\#^{\mathcal{B}\times\mathcal{C}}\mu_n(d\mathbf{b},d\mathbf{c})$$

$$= \int_{\mathcal{B}\times\mathcal{C}}\left(\int_{\mathcal{A}} g(\mathbf{a},\mathbf{b},\mathbf{c})\rho(\mathbf{a},\mathbf{b})\nu(d\mathbf{a})\right)\mathsf{Pr}_\#^{\mathcal{B}\times\mathcal{C}}\mu_0(d\mathbf{b},d\mathbf{c}),$$

where we used disintegration of measures. This completes the proof of the lemma. $\qquad\square$

Appendix E. Proof of Theorem 3.7. Let $g \in U_b(\mathcal{X} \times \mathcal{Y}^{1:N} \times \mathcal{U}^{1:N})$, and define a functional $\tilde{J}$ as

$$\tilde{J}(\lambda^{1:N}) := \int g(\mathbf{x}, \mathbf{y}^{1:N}, \mathbf{u}^{1:N}) \left(\prod_{i=1}^{N} \lambda^i(d\mathbf{u}^i | \mathbf{y}^i) \right) \mathbb{P}\left\{ d\mathbf{x}, d\mathbf{y}^{1:N} \right\},$$

where $\lambda^i \in \wp(\mathcal{U}^i \times \mathcal{Y}^i \times \mathcal{X})$. We proceed with the proof in two steps. In the first step, we show that there exists a subsequence $\{n_k\}_{k \in \mathbb{N}}$ such that $\lim_{k \to \infty} \tilde{J}(\lambda_{n_k}^{1:N}) = \tilde{J}(\lambda_0^{1:N})$. Then, we show that $\lim_{n \to \infty} \tilde{J}(\lambda_n^{1:N}) = \tilde{J}(\lambda_0^{1:N})$ using the first step.

Step 1. Consider the expressions

$$\left| \int g(\mathbf{x}, \mathbf{y}^{1:N}, \mathbf{u}^{1:N}) \left(\prod_{i=1}^{N} \lambda_n^i(d\mathbf{u}^i | \mathbf{y}^i) \right) \mathbb{P}\left\{ d\mathbf{x}, d\mathbf{y}^{1:N} \right\} \right.$$

$$\left. - \int g(\mathbf{x}, \mathbf{y}^{1:N}, \mathbf{u}^{1:N}) \left(\prod_{i=1}^{N} \lambda_0^i(d\mathbf{u}^i | \mathbf{y}^i) \right) \mathbb{P}\left\{ d\mathbf{x}, d\mathbf{y}^{1:N} \right\} \right|$$

$$= \left| \sum_{j=1}^{N} \int g(\mathbf{x}, \mathbf{y}^{1:N}, \mathbf{u}^{1:N}) \left(\prod_{i=1}^{j-1} \lambda_0^i(d\mathbf{u}^i | \mathbf{y}^i) \right) \left(\prod_{i=j+1}^{N} \lambda_n^i(d\mathbf{u}^i | \mathbf{y}^i) \right) \right.$$

$$\left. \left(\lambda_n^j(d\mathbf{u}^j | \mathbf{y}^j) - \lambda_0^j(d\mathbf{u}^j | \mathbf{y}^j) \right) \mathbb{P}\left\{ d\mathbf{x}, d\mathbf{y}^{1:N} \right\} \right|$$

$$\leq \sum_{j=1}^{N} \left| \int g(\mathbf{x}, \mathbf{y}^{1:N}, \mathbf{u}^{1:N}) \left(\prod_{i=1}^{j-1} \lambda_0^i(d\mathbf{u}^i | \mathbf{y}^i) \eta^i(\mathbf{x}, \mathbf{y}^i) \nu_{\mathcal{Y}^i}(d\mathbf{y}^i) \right), \right.$$

(E.1)

$$\left. \left(\prod_{i=j+1}^{N} \lambda_n^i(d\mathbf{u}^i | \mathbf{y}^i) \eta^i(\mathbf{x}, \mathbf{y}^i) \nu_{\mathcal{Y}^i}(d\mathbf{y}^i) \right) \left(\lambda_n^j(d\mathbf{u}^j, d\mathbf{y}^j, d\mathbf{x}) - \lambda_0^j(d\mathbf{u}^j, d\mathbf{y}^j, d\mathbf{x}) \right) \right|,$$

where the integration is taken over the space $\mathcal{X} \times \mathcal{Y}^{1:N} \times \mathcal{U}^{1:N}$. Replace the product terms $\prod_{i=1}^{0}$ and $\prod_{i=N+1}^{N}$ by 1 in those expressions. In the statements of Lemma 3.2 and its corollary and Lemma 3.4, take $\mathcal{B} = \mathcal{U}^i \times \mathcal{Y}^i$ for an appropriate index i, and replace $\mathcal{A}$ by $\mathcal{X}$ and $\mathcal{C}$ by appropriate product spaces.

Applying Corollary 3.3, we conclude that the function

$$\int_{\mathcal{U}^{1:j-1} \times \mathcal{Y}^{1:j-1}} g(\mathbf{x}, \mathbf{y}^{1:N}, \mathbf{u}^{1:N}) \left(\prod_{i=1}^{j-1} \lambda_0^i(d\mathbf{u}^i | \mathbf{y}^i) \eta^i(\mathbf{x}, \mathbf{y}^i) \nu_{\mathcal{Y}^i}(d\mathbf{y}^i) \right)$$

is uniformly continuous in $\mathbf{x}$, $\mathbf{y}^{j:N}$, and $\mathbf{u}^{j:N}$. Next, we use Lemma 3.4 to conclude that the sequence of functions

$$\left\{ \int_{\mathcal{U}^{-j} \times \mathcal{Y}^{-j}} g(\mathbf{x}, \mathbf{y}^{1:N}, \mathbf{u}^{1:N}) \prod_{i=1}^{j-1} \lambda_0^i(d\mathbf{u}^i | \mathbf{y}^i) \eta^i(\mathbf{x}, \mathbf{y}^i) \nu_{\mathcal{Y}^i}(d\mathbf{y}^i) \right.$$

$$\left. \left(\prod_{i=j+1}^{N} \lambda_n^i(d\mathbf{u}^i | \mathbf{y}^i) \eta^i(\mathbf{x}, \mathbf{y}^i) \nu_{\mathcal{Y}^i}(d\mathbf{y}^i) \right) \right\}_{n \in \mathbb{N}}$$

is uniformly equicontinuous and bounded on $\mathcal{X} \times \mathcal{Y}^j \times \mathcal{U}^j$ for every $j \in [N]$. Then, there exists a subsequence $\{n_l\}_{l \in \mathbb{N}}$ by Lemma 3.5 for $j = 1$ such that as $l \to \infty$,

$$\left| \int \left(g(\mathbf{x}, \mathbf{y}^{1:N}, \mathbf{u}^{1:N}) \left(\prod_{i=2}^{N} \lambda_{n_l}^i(d\mathbf{u}^i | \mathbf{y}^i) \eta^i(\mathbf{x}, \mathbf{y}^i) \nu_{\mathcal{Y}^i}(d\mathbf{y}^i) \right) \right) \right.$$

$$\left. \left(\lambda_{n_l}^1(d\mathbf{u}^1, d\mathbf{y}^1, d\mathbf{x}) - \lambda_0^1(d\mathbf{u}^1, d\mathbf{y}^i, d\mathbf{x}) \right) \right| \to 0.$$

Along the sequence $\{n_l\}_{l \in \mathbb{N}}$, there exists a further subsequence $\{n_{l_m}\}_{m \in \mathbb{N}}$ for $j = 2$ such that the second term in the summation in (E.2) goes to zero as $k \to \infty$. Continue this process for $j = 3, \ldots, N$ to arrive at a subsequence $\{n_k\}_{k \in \mathbb{N}}$ such that each component of the sum in (E.2) converges to 0 as $k \to \infty$. Thus, we get $\lim_{k \to \infty} \tilde{J}(\lambda_{n_k}^{1:N}) = \tilde{J}(\lambda_0^{1:N})$.

Step 2. We now claim that $\lim_{n \to \infty} \tilde{J}(\lambda_n^{1:N}) = \tilde{J}(\lambda_0^{1:N})$, which we prove by contradiction. Suppose that $\lim_{n \to \infty} \tilde{J}(\lambda_n^{1:N})$ does not exist or is not equal to $\tilde{J}(\lambda_0^{1:N})$. In this case, there exists an $\epsilon_0 > 0$ and a subsequence $\{n_m\}_{m \in \mathbb{N}}$ such that $|\tilde{J}(\lambda_{n_m}^{1:N}) - \tilde{J}(\lambda_0^{1:N})| > \epsilon_0$ for all $m \in \mathbb{N}$. From Step 1 of the proof, we know that there exists a further subsequence $\{n_{m_k}\}_{k \in \mathbb{N}}$ such that $\lim_{k \to \infty} \tilde{J}(\lambda_{n_{m_k}}^{1:N}) = \tilde{J}(\lambda_0^{1:N})$, which is a contradiction. Thus, $\lim_{n \to \infty} \tilde{J}(\lambda_n^{1:N}) = \tilde{J}(\lambda_0^{1:N})$.

We now arrive at the result of the theorem by [29, Theorem 9.1.5, p. 372]. □

Appendix F. Proof of Lemma 3.10. First, note that the information constraints of the limits λ_0^1 and λ_0^2 are satisfied due to Assumption 3.3 and Lemma 3.6. To establish the result, we follow the same steps as in the proof of Theorem 3.7 in Appendix E above with some minor modifications. Let $g \in U_b(\mathcal{X} \times \mathcal{Y}^{1:2} \times \mathcal{U}^{1:2})$. Consider the following expressions:

$$\left| \int g(\mathbf{x}, \mathbf{y}^{1:2}, \mathbf{u}^{1:2}) \left(\prod_{i=1}^{2} \lambda_n^i(d\mathbf{u}^i | \mathbf{y}^i) \right) \mathbb{P}\{d\mathbf{x}, d\mathbf{y}^{1:2}\} \right.$$

$$\text{(F.1)} \quad \left. - \int g(\mathbf{x}, \mathbf{y}^{1:2}, \mathbf{u}^{1:2}) \left(\prod_{i=1}^{2} \lambda_0^i(d\mathbf{u}^i | \mathbf{y}^i) \right) \mathbb{P}\{d\mathbf{x}, d\mathbf{y}^{1:2}\} \right|$$

$$\leq \left| \int g(\mathbf{x}, \mathbf{y}^{1:2}, \mathbf{u}^{1:2}) \lambda_n^2(d\mathbf{u}^2 | \mathbf{y}^2) \left(\lambda_n^1(d\mathbf{u}^1 | \mathbf{y}^1) - \lambda_0^1(d\mathbf{u}^1 | \mathbf{y}^1) \right) \mathbb{P}\{d\mathbf{x}, d\mathbf{y}^{1:2}\} \right|$$

$$+ \left| \int g(\mathbf{x}, \mathbf{y}^{1:2}, \mathbf{u}^{1:2}) \lambda_0^1(d\mathbf{u}^1 | \mathbf{y}^1) \left(\lambda_n^2(d\mathbf{u}^2 | \mathbf{y}^2) - \lambda_0^2(d\mathbf{u}^2 | \mathbf{y}^2) \right) \mathbb{P}\{d\mathbf{x}, d\mathbf{y}^{1:2}\} \right|,$$

where the integration is taken over the space $\mathcal{X} \times \mathcal{Y}^{1:2} \times \mathcal{U}^{1:2}$. We can now mimic Step 2 of the proof of Theorem 3.7 in Appendix E to complete the proof of the lemma. □

Appendix G. Proof of Lemma 4.3. First, we recall a general version of Markov's inequality.

LEMMA G.1 (generalized Markov's inequality). *Let* $\phi : \mathcal{A} \to \mathbb{R}$ *be a nonnegative measurable function, and let* $\mathcal{A} \subset \mathcal{A}$ *be a Borel measurable set such that* $\inf_{\mathbf{a} \in \mathcal{A}} \phi(\mathbf{a}) > 0$. *Then,* $\mathbb{P}\{\mathcal{A}\} \leq \mathbb{E}[\phi \mathbb{1}_{\mathcal{A}}] / (\inf_{\mathbf{a} \in \mathcal{A}} \phi(\mathbf{a}))$.

Proof. Note that $(\inf_{\mathbf{a} \in \mathcal{A}} \phi(\mathbf{a})) \mathbb{1}_{\mathcal{A}}(\mathbf{a}) \leq \phi(\mathbf{a}) \mathbb{1}_{\mathcal{A}}(\mathbf{a})$ for all $\mathbf{a} \in \mathcal{A}$. Taking expectations on both sides leads us to the result. □

We want to show that the set of measures in $\mathcal{M}$ is tight. Toward this end, we fix $\epsilon > 0$ and show that there exist compact sets $\mathcal{K}_\epsilon \subset \mathcal{A}$ and $\mathcal{L}_\epsilon \subset \mathcal{B}$ such that

$\mathsf{Pr}_{\#}^{\mathcal{A}\times\mathcal{B}}\mu(\mathcal{K}_\epsilon \times \mathcal{L}_\epsilon) = \mu(\mathcal{K}_\epsilon \times \mathcal{L}_\epsilon \times \mathcal{C}) > 1 - 2\epsilon$ for all $\mu \in \mathcal{M}$. This proves that the set of measures $\mathcal{M}$ is tight.

Since the set of measures $\mathcal{N}$ is tight, there exists a compact set $\mathcal{K}_\epsilon \subset \mathcal{A}$ such that $\zeta(\mathcal{K}_\epsilon^{\complement}) < \epsilon$ for all $\zeta \in \mathcal{N}$. Pick $M \in \mathbb{R}^+$ sufficiently large such that $M > k/\epsilon$. We carry out the analysis for the two cases separately.

1. Assume that ϕ satisfies the first condition in Definition 4.2. Given M and $\mathcal{K}_\epsilon$, let $\mathcal{L}_\epsilon \subset \mathcal{B}$ be the compact set such that $\inf_{(\mathbf{a},\mathbf{b},\mathbf{c}) \in (\mathcal{K}_\epsilon \times \mathcal{L}_\epsilon^{\complement} \times \mathcal{C})} \phi(\mathbf{a},\mathbf{b},\mathbf{c}) \geq M$. Now note that $\mu(\mathcal{K}_\epsilon^{\complement} \times \mathcal{B} \times \mathcal{C}) < \epsilon$ for all Borel sets $\mathcal{B} \subset \mathcal{B}$. Let $\mathcal{E} = (\mathcal{K}_\epsilon \times \mathcal{L}_\epsilon)^{\complement} \times \mathcal{C}$ and note that $\mathcal{E} = (\mathcal{K}_\epsilon \times \mathcal{L}_\epsilon^{\complement} \times \mathcal{C}) \bigcup (\mathcal{K}_\epsilon^{\complement} \times \mathcal{B} \times \mathcal{C})$.
   Define $\mathcal{E}_1 = \mathcal{K}_\epsilon \times \mathcal{L}_\epsilon^{\complement} \times \mathcal{C}$ and $\mathcal{E}_2 = (\mathcal{K}_\epsilon^{\complement} \times \mathcal{B} \times \mathcal{C})$. It is easy to verify that $\mathcal{E}_1 \bigcap \mathcal{E}_2 = \emptyset$. We use generalized Markov's inequality (Lemma G.1) to get

$$\mu(\mathcal{K}_\epsilon \times \mathcal{L}_\epsilon^{\complement} \times \mathcal{C}) = \mu(\mathcal{E}_1) \leq \frac{\mathbb{E}\left[\phi 1_{\mathcal{E}_1}\right]}{\inf_{x \in \mathcal{E}_1} \phi(x)} \leq \frac{k}{M} < \epsilon$$

   for all $\mu \in \mathcal{M}$, where $1_{\mathcal{E}_1}$ is the indicator function over the set $\mathcal{E}_1$. By the additivity property of probability measures, we get $\mu(\mathcal{E}) = \mu(\mathcal{E}_1) + \mu(\mathcal{E}_2) < 2\epsilon$.

2. Now suppose that ϕ satisfies the second assumption. For every $a \in \mathcal{K}_\epsilon$, let $\mathcal{O}_a \subset \mathcal{A}$ be the open neighborhood of $a \in \mathcal{K}_\epsilon$ and $\mathcal{L}_\mathbf{a} \subset \mathcal{B}$ be the compact set in $\mathcal{B}$ such that $\inf_{(\mathbf{a},\mathbf{b},\mathbf{c}) \in (\mathcal{O}_a \times \mathcal{L}_a^{\complement} \times \mathcal{C})} \phi(\mathbf{a},\mathbf{b},\mathbf{c}) \geq M$. Notice that $\{\mathcal{O}_\mathbf{a}\}_{\mathbf{a} \in \mathcal{K}_\epsilon}$ is an open cover for $\mathcal{K}_\epsilon$. One can now use the definition of compactness and arguments as in part 1 with some modifications to prove that part 1 holds.

Note that $\mathcal{K}_\epsilon \times \mathcal{L}_\epsilon$ is a compact set in $\mathcal{A} \times \mathcal{B}$ and its complement has small measure. Thus, we conclude that the set of probability measures $\mathsf{Pr}_{\#}^{\mathcal{A}\times\mathcal{B}}\mathcal{M}$ is tight.

If ϕ is lower semicontinuous, then $\mu \mapsto \int \phi\, d\mu$ is a lower semicontinuous functional [31, Lemma 4.3]. Thus, we conclude that $\mathcal{M}$ is in fact weak* closed, and therefore $\mathcal{M}$ is weak* compact. $\quad\square$

Appendix H. Proof of Lemma 5.7. Assumption 2.2 and the structure of the cost function of the team as defined in (5.3) imply that for any $i \in [N]$ and $t \in [T]$, we have

$$\int_{\Omega_0 \times \mathcal{Y}_{1:t-1}^{1:N} \times \mathcal{U}_{1:t-1}^{1:N} \times \mathcal{Y}_t^i \times \mathcal{U}_t^i} \bar{c}_t^i(\mathbf{u}_t^i, \omega_0, \mathbf{u}_{1:t-1}^{1:N}, \mathbf{y}_{1:t-1}^{1:N}, \mathbf{y}_t^i)\, d\lambda_{1:t-1}^{1:N}\, d\lambda_t^i\, \mathbb{P}\{d\omega_0\} \leq J(\tilde{\pi}_{1:T}^{1:N})$$

for any choice of $\lambda_{1:T}^{1:N} \in \mathcal{N}_{1:T}^{1:N}$. Also recall from Lemma 5.6 that $\bar{c}_t^i$ is in class $\mathrm{IC}(\Omega_0 \times \mathcal{U}_{1:t-1}^{1:N} \times \mathcal{Y}_{1:t-1}^{1:N} \times \mathcal{Y}_t^i, \mathcal{U}_t^i)$. We now use the result of Lemma 4.3 and the principle of mathematical induction to prove the result.

Step 1. Consider agent $(1,1)$. We know that $\bar{c}_1^1$ is a nonnegative continuous function in class $\mathrm{IC}(\Omega_0 \times \mathcal{Y}_1^1, \mathcal{U}_1^1)$. Moreover, the measure on $\Omega_0 \times \mathcal{Y}_1^1$ is tight. Using the result of Lemma 4.3, we get that $\mathcal{N}_1^1$ is a tight set of measures.

Step 2. Using the same argument as in Step 1, we conclude that $\mathcal{M}_1^i$ is a tight set of measures for all $i \in [N]$.

Step 3. Let us assume that $\mathcal{M}_s^i$ is a tight set of measures for all $i \in [N]$ and $1 \leq s \leq t-1$. Consider any agent (i,t). We know that $\bar{c}_t^i$ is a nonnegative continuous function in class $\mathrm{IC}(\Omega_0 \times \mathcal{U}_{1:t-1}^{1:N} \times \mathcal{Y}_{1:t-1}^{1:N} \times \mathcal{Y}_t^i, \mathcal{U}_t^i)$. Moreover, the set of all possible measures on $\Omega_0 \times \mathcal{U}_{1:t-1}^{1:N} \times \mathcal{Y}_{1:t-1}^{1:N}$ induced by $\mathcal{N}_{1:t-1}^{1:N}$ is tight because $\mathcal{M}_s^i$ is tight for all $i \in [N]$ and $1 \leq s \leq t-1$ by the induction hypothesis. Therefore, using the result of Lemma 4.3, we get that $\mathcal{M}_t^i$ is a tight set of measures.

This completes the induction step, and thus $\mathcal{M}_t^i$ is tight for all $i \in [N]$ and $t \in [T]$.

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