Optimal causal quantization of Markov Sources with distortion constraints

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***Abstract*— For Markov sources, the structure of optimal causal encoders minimizing the total communication rate subject to a mean-square distortion constraint is studied. The class of sources considered lives in a continuous alphabet, and the encoder is allowed to be variable-rate. Both the finite-horizon and the infinite-horizon problems are considered. In the finite-horizon case, the problem is non-convex, whereas in the infinite-horizon case the problem can be convexified under certain assumptions. For a finite horizon problem, the optimal deterministic causal encoder for a kth-order Markov source uses only the most recent k source symbols and the information available at the receiver, whereas the optimal causal coder for a memoryless source is memoryless. For the infinite-horizon problem, a convex-analytic approach is adopted. Randomized stationary quantizers are suboptimal in the absence of common randomness between the encoder and the decoder. If there is common randomness, the optimal quantizer requires the randomization of at most two deterministic quantizers. In the absence of common randomness, the optimal quantizer is non-stationary and a recurrence-based time-sharing of two deterministic quantizers is optimal. A linear source driven by Gaussian noise is considered. If the process is stable, innovation coding is almost optimal at high-rates, whereas if the source is unstable, then even a high-rate time-invariant innovation coding scheme leads to an unstable estimation process.**

I. Introduction

In real-time applications such as remote control of time-sensitive processes, live streaming and voice over Internet, causality in encoding and decoding is a natural limitation. Causal coding has been studied in different contexts and with different assumptions on the classes of sources and encoder types.

A major result on the topic due to Neuhoff and Gilbert [12] establishes that the optimal memoryless coder is the optimal causal encoder, minimizing the data rate subject to a distortion constraint. It is assumed that the sequence $\{x_i\}$ is i.i.d., and evolves on a discrete alphabet. If the source is kth-order Markov, then the optimal causal fixed-rate coder minimizing any measurable distortion uses only the last k source symbols, together with the current state at the receiver's memory [16]. The results of [16] were extended in [14] to systems with noisy feedback, under the assumption of a fixed decoder structure with finite memory. High-rate encoding of a stable stationary process is treated in [9], where a memoryless quantizer followed by a conditional entropy coder is found to be at most 0.25 bits worse than any noncausal encoder in the limit of low distortion.

If one allows variable rate coding, the optimization problem for the evaluation of the optimal quantizer becomes an infinite-dimensional problem [7]. Entropy, as a lower bound on error-free transmission for a discrete source, can be attained fairly closely by means of entropy coding techniques or via block coding. We note that in variable rate coding, the delays in the transmission of higher length codes might affect the overall delay. Nonetheless, in variable-rate coding, the system is still causal in that no future data is allowed to be used in encoding and decoding. György et al [7], [8] have looked at the problem of *entropy-constrained* quantization for the design of variable rate quantizers for i.i.d sources also obtaining quantitative results for a uniform source, in particular exhibiting the benefit of time-sharing. The results of [16] do not immediately extend to the cases when there is a randomization at the encoder [6]. We will observe that randomization might help in the present setting due to the constraint in the optimization.

Stochastic control can provide a useful tool in the characterization of optimal coding schemes, as has been exhibited by Witsenhausen [16]. An important work in this direction is Borkar [4], which considers infinite horizon quantization of a partially observed source generated by a stationary stable Markov process, allowing variable rate coding. Borkar establishes the existence of an optimal policy over the class of stationary policies. In the present paper, the convex analytic approach to MDPs[1] is adopted to establish optimality of a stationary policy over all admissible policies.

II. Preliminaries and Problem Statement

We first recall the notion of a quantizer.

Definition 2.1: A *quantizer* for a scalar continuous variable is a Borel-measurable mapping Q from the real line to a finite or countable set, characterized by corresponding bins $\{\mathcal{B}_i\}$ and their reconstruction levels $\{q^i\}$, such that $\forall i$, $Q(x) = q^i$ if and only if $x \in \mathcal{B}_i$. $\diamond$

We assume that the quantization bins are regular [7]. Thus, (for scalar quantization) $\mathcal{B}_i$ can be taken to be nonoverlapping semiopen intervals, $\mathcal{B}_i = [\delta_i, \delta_{i+1})$, with "bin edges" $\delta_i <$

[1] An excellent reference is [3], see also Chapter 9 of [2]

$\delta_{i+1}, i = 0, \pm 1, \pm 2, \ldots$. We assume that $q^i \in \mathcal{B}_i$, and that δ_0 is the bin edge that is closest to the origin.

In a dynamic, discrete-time setting, the construction of a quantizer at any time t can depend on past quantizer values. To make this precise, let $\mathcal{X}$ denote the input space, $\hat{\mathcal{X}}$ the output space, and $\mathcal{R}_t = \{q_t\}$ the set of quantizer reconstruction values at time $t = 1, 2, \ldots$. Then, the quantizer at time t, to be denoted by f_t, is a mapping from $\mathcal{X}^t$ to $\mathcal{R}_t$, where $\mathcal{X}^t$ is the t-product of $\mathcal{X}$. Such a quantizer is said to be *dynamic* and *causal*. A quantizer is assumed to be followed by an encoder, which provides the binary representation of the quantization outputs. However, hereafter, we will use the term encoder to refer to the ensemble of the quantizer and the encoder. Finally, we let $g_t : \mathcal{R}_1 \times \mathcal{R}_2 \times \ldots \mathcal{R}_t \to \hat{\mathcal{X}}$ denote the *decoder function*, which again has causal access to the past received values.

The class of quantizers we have introduced above are so-called deterministic causal quantizers, in the sense that for each fixed t, and given f_t and history $\mathcal{H}_t := \{x_s, f_s; s = 1, \ldots, t\}$, the quantizer $f_t(\mathcal{H}_t)$ is a uniquely defined element of $\mathcal{Q}$, where $\mathcal{Q}$ is the space of such deterministic quantizers, and let $\sigma(\mathcal{Q})$ be its σ-algebra. *Randomized quantizers* are a more general class that assign a probability measure to a selection of bins for each fixed $\mathcal{H}_t \in \mathcal{H}^\infty$. More precisely, and by a slight abuse of notation, quantizer policy $q(.|h)$ is randomized if, for each $\mathcal{H}_t \in \mathcal{H}^\infty$, $q(.|\mathcal{H}_t)$ is a probability measure on $\sigma(\mathcal{Q})$, and if, for every fixed $D \in \sigma(\mathcal{Q})$, $q(D|.)$ is a well-defined function on $\mathcal{H}^\infty$, whose restriction to the interval $[1, t]$ agrees with a deterministic quantizer f_t.

A randomized *stationary* quantizer assigns a probability measure to selection of bins for each fixed $x_1^\infty \in \mathcal{X}^\infty$. Let $\mathcal{P}$ denote the set of conditional probability densities on the Borel field on the real line $\mathcal{B}(R)$. A randomized stationary quantizer defines for each t a random variable taking values in $\mathcal{P}$, denoted $p(x_t|f_1^{t-1}, q_1^{t-1})$. A randomized stationary quantizer policy has the property,

$$p(.|\pi_t = \pi) = p(.|h_t, \pi_t = \pi), \quad a.s. \quad \forall h_t(w)$$

for each $\pi \in \mathcal{P}$, with $h_t(w)$ denoting the sample paths of the history process. It is assumed that for each $\pi \in \mathcal{P}$, $q(.|\pi)$ is a probability measure on $\sigma(\mathcal{P})$, for every fixed $D \in \sigma(\mathcal{P})$, $q(D|.)$ is a measurable function on $\mathcal{Q}$. For a deterministic stationary quantizer, $p(.|\pi)$ is a dirac measure.

Let $v_i^m = (v_i, \ldots, v_m)^T$ denote the vector composed of the ith through mth components of an n dimensional vector v, with $i \le m \le n$. Consider a sequence $\{x_t \in R, t = 1, 2, \ldots, N\}$, causally encoded,

$$f_t(x_1^t), \quad 1 \le t \le N,$$

and suppose that a delayless causal decoder generates the estimates,

$$\hat{x}_t = g_t(f_1(x_1), \ldots, f_t(x_1^t)) = g_t(f_1^t, q_1^t),$$

for $1 \le t \le N$. We have two assumptions, one of which is on the set of initial distributions.

Assumption A: The initial state x_0 with distribution p_0 has finite second moment, i.e. $E_{p_0}[x_0^2] < \infty$. ⋄

The second set of assumptions is related to the transition operator of the Markov process, hence the dynamics of the source.

Assumption B:

1) The transition probabilities are μ-irreducible, where μ is Lebesgue measure: $P(x_t \in \mathcal{C}|x_{t-1}) > 0$ for every set $\mathcal{C} \subset R$ with positive Lebesgue measure, and hence for every non-empty open set.
2) The Markov process $\{x_t\}$ forms a positive recurrent Markov chain.
3) The invariant limiting distribution $p_\infty := \lim_{t\to\infty} p_t$, has finite second moment. Furthermore, for every t, $(p_t)\log_2(p_t)$ and x_t^2 are uniformly integrable.
4) For each t, the marginal density function p_t is uniformly continuous.

⋄

Assumption B can be further relaxed, but for convenience in the discussion we will assume it to hold. We will use irreducibility to ensure recurrence properties of the conditional density process. Uniform integrability conditions are needed for the continuity-compactness arguments that will be employed in a weak sense for the existence of optimal quantizers. These all can be significantly relaxed for sources which take values in a finite space.

For instance, the family of stable linear systems driven by Gaussian noise satisfies all these conditions. More generally, uniform integrability holds under a certain stochastic drift condition by the V-uniform ergodic theorem of [10].

Let $H(f_1^N)$ denote, with a slight abuse of notation, the entropy of the quantizer outputs $q_1^N := \{q_1, \ldots, q_N\}$ under the quantization policies $f_i(\cdot),\ i = 1, \ldots, N$. That is, with $p_t(q_i)$ denoting the probability of q_i at time t:

$$p_t(q_i) := p(x_t \in \mathcal{B}_i | f_1^{t-1}, q_1^{t-1})$$

let

$$H(f_1^N) = \sum_{t=1}^{N} H(f_t|f_1^{t-1}) = -\sum_{t=1}^{N} E[p_t(q)\log_2 p_t(q)]$$
$$q_i = f(x_1^i)\,,\ i = 1, \ldots, N\,.$$

We study the following constrained minimization problem: For a given positive integer N,

$$\inf_{f_1^N} \frac{1}{N} H(f_1^N) \tag{1}$$

subject to

$$\frac{1}{N}\sum_{t=1}^{N} E[(x_t - \hat{x}_t(f_1^t, q_1^t))^2] \le D, \tag{2}$$

for some finite $D > 0$, where $\hat{x}_t(f_1^t)$ is (as the output, $g_t(\cdot)$, of the decoder) the conditional mean of x_t given the quantizer policy $\{f_s,\ s \le t\}$, and the output of the quantizer, $q_s,\ s \le t$. When clear, we will drop the notation $\hat{x}_t(f_1^t, q_1^t)$ and use $\hat{x}_t(f_1^t)$ instead. We will consider also the infinite-horizon case, when $N \to \infty$. Further, we will study the improvement in the value of (1) when randomization is allowed on quantizer policies.

III. Finite-Horizon Problem

In the causal coding literature, the underlying optimization problem has generally been restricted to finite dimensional spaces. The analysis then builds on the fact that a continuous function over a compact set attains a minimum. However, when there is an entropy constraint, as opposed to a fixed length rate constraint, the optimization problem is one of infinite dimension and the optimal quantizer could then have infinitely many quantization levels [8]. The appropriate framework in this case is infinite-dimensional optimization and the weak topology.

Lemma 3.1: There exists a solution to the optimization problem.

The following is a known result for certain special cases, in particular had been studied in [12] and also [15]. The following follows the observation that the constraint problem can be posed as an unconstrained problem by introducing a single Lagrange multiplier. Consideration of the first order optimality condition, for each of the quantizers used in the time stages $t = 1, 2, \ldots, T$, leads to the following result:

Theorem 3.1: Suppose $\{x_t\}$ are i.i.d. random variables which can be discrete or continuous valued. Then, the optimal deterministic encoder uses only the current symbol, and the quantizer is only a function of the marginal distribution of $p(x_t)$.

The above result can be extended to Markovian sources. Witsenhausen [16] studied this problem first with finite alphabet sources, with fixed-length codes. We observe that this can be extended to variable-length codes.

Theorem 3.2: For a kth-order Markov source, the finite-horizon optimal causal deterministic encoder at stage t, $0 \leq t \leq N-1$ use the most recent (available) $\min(t,k)$ symbols and the information available at the receiver. The optimal deterministic encoder for the last stage, N, uses only the distribution of the last symbol and the information available at the receiver.

IV. Infinite-Horizon Solution

In this section we consider the infinite-horizon problem. Recall the following definition.

Definition 4.1: A probability measure η is invariant on $(R, \mathcal{B}(R))$, with $\mathcal{B}(R)$ denoting the Borel subsets of R, if

$$\eta(D) = \int_X P(D|x)\eta(dx), \quad \forall D \in \mathcal{B}(R),$$

where $P(D|x)$ is the transition probability, $P(x_{t+1} \in D | x_t = x)$.

To ease the technical burden, we use the following assumption to *atomize* the state space where the coders live:

Assumption C: The filter at the decoder has arbitrarily large, but finite memory: There exists $d \geq 1$ such that $p(x_t|f_1^{t-1}, q_1^{t-1}) = p(x_t|f_{t-d}^{t-1}, q_{t-d}^{t-1})$. $\diamond$

The following is useful for obtaining the optimality of stationary policies over all admissible policies.

Assumption D: For all quantizers in $\mathcal{Q}$, there exists an arbitrarily small common bin $\mathcal{B}^* = [\delta_a^*, \delta_b^*)$, $\delta_b^* > \delta_a^*$. $\diamond$

Under Assumption B, Markov processes admit an invariant distribution. Furthermore, it follows that, under Assumption B and C, the state distributions at each time t belong to a weak* compact set, which implies tightness, and is equivalent to it under closedness. This means that, for every ϵ, there exists a compact set $\mathcal{K}_\epsilon \subset R$, such that $P_t(x_t \in \mathcal{K}_\epsilon) > 1 - \epsilon$.

Hence if $\mathcal{B}^*$ is sufficiently far away from the origin, then since as $p \to 0, p\log_2(p) \to 0$, with $P(x \in \mathcal{B}^*)$ small, Assumption D above does not impact the entropy cost of the quantizers significantly. We will use the common small bin to generate a neighborhood of a recurrent state under every policy.

The problem we are interested in is the minimization of the quantity

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} H(f_t|f_1^{t-1}) \tag{3}$$

subject to the average distortion constraint

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} E[(x_t - \hat{x}_t(f_1^t))^2] \leq D. \tag{4}$$

The state process is partially observed by the decoder. This class of infinite horizon problems is called Partially Observed Markov Decision Process (POMDP) problems. As is typical in MDP problems, the partially observed Markov chain can be converted to a Fully Observed Markov Decision Process (FOMDP), by enlarging the state space, and replacing the state with the belief on the state [4]. Properties of conditional probability leads to the following expression for $P(x_n|f_1^{n-1})$:

$$\frac{\int P(x_{n-1}|f_1^{n-2})P(f_{n-1}|x_{n-1})P(x_n|x_{n-1})dx_{n-1}}{\int\int P(x_{n-1}|f_1^{n-2})P(f_{n-1}|x_{n-1})P(x_n|x_{n-1})dx_n dx_{n-1}}. \tag{5}$$

The entropy and the distortion constraint can be written as a function of this conditional density for all time stages. Following the notation of [4], we define

$$\pi_n(x) := P(x_n = x|f_1^{n-1}).$$

Let $\mathcal{P}$ be the set of probability distributions for $\pi_n(x), n \geq 1$. Then the conditional density and the quantization output process, $(\pi_n(x), f_n)$, form a joint Markov process in $\mathcal{P} \times \mathcal{Q}$.

We can extend the definition of Definition 4.1 to the extended state as follows.

Definition 4.2: A probability measure η is invariant on $(\mathcal{P} \times \mathcal{Q}, \sigma(\mathcal{P} \times \mathcal{Q}))$, if

$$\eta(D) = \int_{\mathcal{P}\times\mathcal{Q}} p_f(D|\pi)\eta(\pi, f), \quad \forall D \in \sigma(\mathcal{P} \times \mathcal{Q}),$$

where $p_f(D|\pi)$ is the transition probability under f, $p(\pi_{t+1} \in D|\pi_t = \pi, f_t = f)$.

Let Γ denote the steady state distribution of $\phi_t = (f_t, \pi_t)$. We then have the minimization of

$$(\Gamma, H) \tag{6}$$

subject to

$$(\Gamma, C) \leq D \tag{7}$$

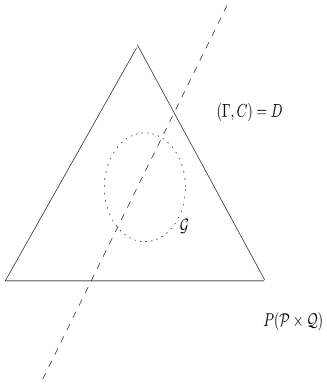

Fig. 1: The triangular region denotes the space of probability distributions. The convex region is the set $\mathcal{G}$, where the boundary of $\mathcal{G}$ corresponds to occupation measures generated by the deterministic quantizers. The dashed line denotes the linear constraint due to the distortion constraint. The optimal solution will live on the boundary defined by the intersection of $\mathcal{G}$ and the dashed line. Under Assumption D, any point on this boundary can be attained by the randomization of two deterministic quantizers.

over all admissible quantizers, where $C(\phi)$ is the conditional distortion, and $H(\phi)$ is the entropy of the quantizer applied to the conditional density and $(.,.)$ denotes the inner product. This is an infinite dimensional linear program, and under general conditions, there exists a solution. In the following we present a result toward this direction.

A. The structure of optimal randomized quantizers

Theorem 4.1: Suppose for a Markov process, Assumptions A and B hold. Then there exists an optimal stationary deterministic quantizer solving (6)-(7).

We say a quantizer is admissible if f_t is a causal quantizer, that is the policy is measurable with respect to the past information generated as defined previously.

Lemma 4.1: The set of occupation measures for the state process (belief process) is a closed, convex set.

Theorem 4.2: If under any two quantizers, the set of corresponding occupation measures for the state are such that for some $B \subset \mathcal{P}$, $\eta_\pi^f(B) > 0$ and $\eta_\pi^{f'}(B) > 0$, then there exists a stationary quantizer which is as good as the optimal admissible quantizer minimizing (6)-(7). In particular, under Assumption D, the above argument holds.

Proof: The set of occupation measures for both the state and the control is also closed and convex, which is also tight. Since the cost function is weakly continuous in the input distribution and control, there exists an optimal occupation measure. One needs to show that this optimal occupation measure is achieved by some stationary quantizer from any point π in the invariant set, and follows the arguments in [3]. $\diamond$

Theorem 4.3: For both the finite-horizon and the infinite-horizon problems, performance can be improved using a randomized causal quantizer. The optimal randomized stationary policies are convex combinations of at most two stationary deterministic policies, with the randomization information at the encoder made available to the decoder.

Proof: The fact that only two-point randomization is needed follows from the fact that there is only one constraint in the optimization problem (see [1] or [2]), which can be regarded as an infinite dimensional static optimization problem. $\diamond$

It is impractical to implement randomized quantizers since these require common randomness between the encoder and the decoder and lack of such information increases the entropy due to the synchronization error at the encoder and the decoder. However, for the infinite-horizon problem, one might achieve the optimal performance without the assumption of such common randomness. This can be achieved via time-sharing, which has to exploit the recurrence properties of Markov chains. In the following we investigate this approach. In this section we provide the most general solution to the optimization problem, via an analysis of non-stationary quantizers. We adopt an approach presented by Ross [13] via *past-dependent* coding policies. We shall switch quantizer policies at certain visits to a particular state. The set of probability density functions, however, unlike discrete probability distribution functions requires a more involved analysis for recurrence properties. However, due to the Assumption C above, this state process will live in an atomic space.

Theorem 4.4: Suppose that there exists a state $\pi^*(x)$, which is visited infinitely often under each of the deterministic policies used in the randomized stationary quantizer. Then, there exists an optimal time-sharing scheme achieving the performance of the optimal occupation measure. In particular, under Assumption D, the argument above holds.

Proof: There is an optimal occupation measure. Now suppose, this measure is a convex combination of two deterministic policies with randomization rate $\eta = m/n$, m, n integers. By assumption under both policies, the expected excursion to the state from itself is finite. Now, apply the policy f' in the first m cycles between the visits to the recurrent state C, and apply f'' in the remaining $n - m$ successive visits to the same state. $\diamond$

Remark For the memoryless discrete source case, it was shown in [12] that the optimal memoryless encoder time-shares between two scalar quantizers. A result similiar in spirit to the one by [12], is applicable to dynamic encoding as well, with the difference here being in the additional analysis required in the recurrence properties of the chain. In such a case, the policy is not stationary. $\diamond$

B. Asymptotic Performance of Innovation Coding

Consider a system generated by the following dynamics:

$$x_{t+1} = ax_t + w_t \tag{8}$$

where x_t is the state at time t, and $\{w_t\}$ is a sequence of zero-mean independent, identically distributed (i.i.d.) Gaussian random variables. We first consider the case where $\mu < 0$ and hence $|a| < 1$. For the coding problem, there does exist a solution by our earlier argument due to the following. We first argue that there exists an invariant density for the error process.

Define $\xi_t := (x_t - E[x_t|q_1^t])$. First, observe that the joint process $\{e_t, \xi_t\}$ is Markov if the quantizer is time invariant. Hence, we have the following dynamics:

$$e_{t+1} = a\xi_t + w_t, \quad \xi_{t+1} = a\xi_t - Q(e_t) + w_t \qquad (9)$$

Hence, by the Comparison theorem [10], there exists an $\epsilon > 0$ and a bounded *petite* set $C = \{(e, \xi) : x^2 + \xi^2 \leq 2E[w_0^2/(1-a^2)]\}$ such that (see [10])

$$E[e_{t+1}^2 + \xi_{t+1}^2|e_t, \xi_t] \leq (1-\epsilon)(e_t^2 + \xi_t^2) + E[w_0^2]1_{e_t \in C} ,$$

where $1_{(.)}$ is the indicator function, and $\epsilon = 1 - a^2$. This ensures that there exists an invariant distribution for the Markovian joint process $\{e_t, \xi_t\}$, and hence for the marginal e_t as well.

We now consider a scheme where the innovation process,

$$e_t = x_t - aE[x_{t-1}|q_1^{t-1}],$$

is quantized at each time stage. The proof of the result below uses the entropy power inequality, and it has a close connection with the application of the Shannon lower bound as was first used by Linder and Zamir [9] in the context of causal coding.

Theorem 4.5: For causal coding of the source given in (8) with stable dynamics, a lower bound on the average rate R, subject to an average distortion constraint D, is given by:

$$R(D) \geq (1/2)\log(a^2 + \frac{\sigma^2}{D})$$

We further have the following:

Theorem 4.6: For causal coding of the source given in (8) with stable dynamics, suppose innovations are uniformly quantized to obtain a stationary error process. In the limit of low distortion, such an encoder is at most 0.254 bits worse than any (possibly noncausal) encoder.

Theorem 4.7: Consider causal coding of the source given in (8) with unstable dynamics ($|a| > 1$). The use of a time-invariant innovation quantizer leads to a transient Markov chain, no matter how high the data rate is so long as it is finite. Hence, $\lim_{t\to\infty} e_t = \infty$ almost surely.

V. Conclusion

Optimality of stationary quantizers under common randomization information between the encoder and the decoders over all admissible quantizers is established. If common randomness is not available, then the optimal quantizer is a past-dependent, non-stationary, recurrence based policy and there is no loss due to the absence of common randomness. Further numerical examples are currently under investigation.

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