An Asymptotically Optimal Two-Part Coding Scheme for Networked Control under Fixed-Rate Constraints

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***Abstract*— It is known that fixed rate adaptive quantizers can be used to stabilize an open-loop-unstable linear system driven by unbounded noise. These quantizers can be designed so that they have near-optimal rate, and the resulting system will be stable in the sense of having an invariant probability measure, or ergodicity, as well as the boundedness of the state second moment. However, results on the minimization of the state second moment for such quantizers, an important goal in practice, do not seem to be available. In this paper, we construct a two-part adaptive coding scheme that is asymptotically optimal in terms of the second moments as the data rate grows large. The first part, as in prior work, leads to ergodicity (via positive Harris recurrence) and the second part attains order optimality of the invariant second moment, resulting in near optimal performance at high rates.**

I. INTRODUCTION

We consider the linear system

$$x_{t+1} = ax_t + bu_t + d_t, \tag{1}$$

where $|a| \geq 1$, $b \neq 0$, and d_t is a sequence of i.i.d. Gaussian random variables $d_t \sim \mathcal{N}(0, \sigma^2)$. The state variable x_t is driven by the noise d_t and the aim is to control the state through the action u_t. The system is open-loop-unstable but is stabilizable.

First suppose that the system is fully observed. If one minimizes the average quadratic cost of the state, $E\left[x_t^2\right]$, over any time horizon (with no penalty on control) then the optimal control is $u_t = -\frac{a}{b}x_t$ and the optimal cost is σ^2 (via a Riccati equation optimality argument with no cost on control [1]). In contrast, here we assume that the controller only has access to x_t through a discrete noiseless channel of capacity C bits.

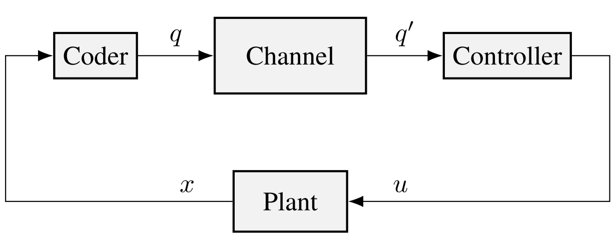

Fig. 1. Block diagram of the communication and control loop.

Thus, the data rate is fixed, and we assume zero coding delay. In this setup, it becomes necessary to describe not just a control policy, but also a coding scheme with which to communicate information about the current state variable.

For systems of this nature, various authors have obtained the smallest channel capacity above which stabilization is possible, under various assumptions on the system and the admissible coders and controllers. This result is usually referred to as a *data-rate theorem*, and in the scalar case (1) we consider, it reduces to simply $C > \log|a|$. Some of the earliest works in this context are [2] and [3]. More general versions of the data-rate theorem have been proven in [4] and [5]. For noisy systems and mean-square stabilization, or more generally, moment-stabilization, analogous data-rate theorems have been proven in [6] and [7], see also [8], [9].

In [10], [11], a joint fixed-rate coding and control scheme is given which stabilizes the system (1) while being nearly rate-optimal, in that the rate used satisfies only $C > \log(|a|+1)$. This is achieved using an adaptive uniform quantization scheme, where the quantizer bin sizes "zoom" in and out exponentially to track the state x_t. Here, the notion of stability is ergodicity and finiteness of all limiting system moments. By increasing a sampling period T the achievable rate $\frac{1}{T}\log(|a|^T + 1)$ gets arbitrarily close to $C > \log|a|$ [12, Theorem 2.3]. The same result also applies to multi-dimensional systems [12]. Furthermore, this scheme leads to a closed loop system which is positive Harris recurrent, and hence, ergodic. For related recent fixed-rate constructions, we refer the reader to [13] and [14]. We also note that there has been a large literature on optimal networked control, see [15], [16], [17], [18], [19], [20], [21], [22], [23], [24].

However, despite being near rate-optimal for achieving stability (i.e., *finite* system moments), the scheme in [10], [11] has not been shown to be asymptotically second-moment optimal as the data rate grows large.

In this paper, we present a variation on the coding and control scheme used in [10], [11] by including an additional quantization stage, which is fixed in time, unlike the first stage of the coding scheme. Crucially utilizing the ergodicity results of the first coding stage, we show that this two-part coding scheme attains second-moment optimality with near optimal rate of convergence. While multi-stage quantization schemes have been studied before in the source coding literature [25], our implementation is novel in that one stage of the code is time-adaptive and stabilizing. An inspiration for this approach also comes to us from Berger [26] and Sahai [27].

In this paper, we show that this new scheme retains key stability and ergodicity properties ensured by the original one-stage scheme, closely appealing to the existing arguments made in [10], [11]. In our main results, Theorems 3.4 and 3.6, we show that the new scheme is asymptotically second-moment optimal, i.e., as the data rate goes to infinity, the system second moment converges to the optimum σ^2 at a rate which is arbitrarily close to order-optimal in a polynomial sense. We also illustrate this convergence by numerical results.

II. PRELIMINARIES

A. Quantization

To communicate over a finite capacity channel, it is necessary to employ quantization schemes. We will work with the following class of quantizers, as introduced in [10], [11]. For a given bin size $\Delta > 0$ and even number of bins $M \geq 2$, we define the *modified uniform quantizer* Q_M^Δ by,

$$Q_M^\Delta(x) = \begin{cases} \Delta \left\lfloor \frac{x}{\Delta} \right\rfloor + \frac{\Delta}{2}, & \text{if } x \in \left[-\frac{M}{2}\Delta, \frac{M}{2}\Delta\right) \\ \frac{M}{2}\Delta - \frac{\Delta}{2}, & \text{if } x = \frac{M}{2}\Delta \\ 0, & \text{if } |x| > \frac{M}{2}\Delta. \end{cases} \tag{2}$$

This quantizer uniformly quantizes $x \in \left[-\frac{M}{2}\Delta, \frac{M}{2}\Delta\right]$ into M bins of size Δ and maps all larger x to zero. This requires $M+1$ output levels.

This quantizer is almost as simple as possible (aside from the overload symbol, this is typical uniform quantization), and its use leads to stability of the closed-loop process with near rate-optimality, as discussed in Section I.

We will consider two primary applications of this class of quantizers. First, we consider *adaptive* quantizers, whose bin sizes vary with time. Fix an even number of bins $K \geq 2$ and let $\{\Delta_t\}_{t=0}^\infty$ be a sequence of strictly positive bin sizes. We will then make use of the adaptive modified uniform quantizer $Q_K^{\Delta_t}$.

Secondly, we consider quantizers fixed in time, whose bin size is a function of the number of bins. For a given (even) number of bins $N \geq 2$, we choose a bin size $\Delta_{(N)}$ and consider the quantizer $Q_N^{\Delta_{(N)}}$. To distinguish this from the adaptive case, we denote $U_N := Q_N^{\Delta_{(N)}}$. We also allow for $N = 0$, for which we write $U_0 \equiv 0$.

B. System Description

Recall the linear system (1) with the information constraints of Figure 1. Since the system is open-loop-unstable, any fixed quantization policy will make the system transient. We will consider a two-part coding scheme, where the first part is adaptive and the second is fixed. The adaptive part will yield stability, and the fixed part will yield an optimal rate of convergence.

Suppose $\{\Delta_t\}_{t=0}^\infty$ is a sequence such that Δ_{t+1} is a function of only Δ_t and the indicator random variable $\mathbf{1}_{\{|x_t| \leq \frac{K}{2}\Delta_t\}}$. Also assume that both the encoder and the decoder (controller) know Δ_0. Then so long as $Q_K^{\Delta_t}(x_t)$ is sent over the channel, it is possible to synchronize knowledge of Δ_t between the quantizer and the controller, since $|x_t| \leq \frac{K}{2}\Delta_t$ if and only if $Q_K^{\Delta_t}(x_t) \neq 0$.

We now describe the proposed communication and control scheme. For $\{\Delta_t\}_{t=0}^\infty$ as above, we calculate the adaptive quantizer output $Q_K^{\Delta_t}(x_t)$ and the adaptive system error $e_t := x_t - Q_K^{\Delta_t}(x_t)$. Then, using a fixed (i.e., non-adaptive) quantizer U_N with bin size $\Delta_{(N)}$ as in Section II-A, we calculate the fixed quantizer output $U_N(e_t)$. We then send $Q_K^{\Delta_t}(x_t)$ and $U_N(e_t)$ across the noiseless channel using a total of $C = \log_2(K+1) + \log_2(N+1)$ bits.

The controller then applies the control given by

$$u_t = -\frac{a}{b}\left(Q_K^{\Delta_t}(x_t) + U_N(e_t)\right).$$

This coding and control scheme is illustrated below.

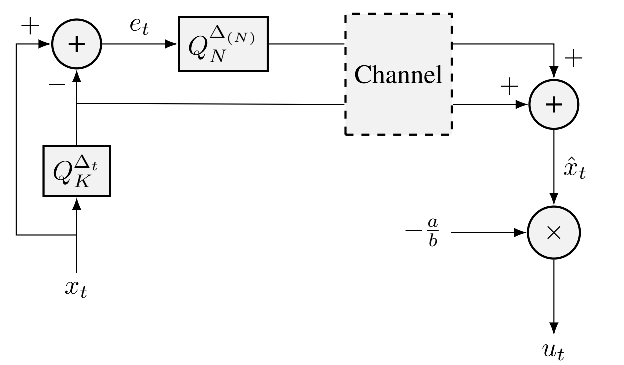

Fig. 2. Block diagram of the two-stage coding and control scheme.

We note that the term $U_N(e_t)$ distinguishes our control scheme from [10], [11]. This control is chosen to mirror the optimal fully observed control, where $Q_K^{\Delta_t}(x_t) + U_N(e_t)$ is a good estimate of the true state x_t. By (1), this results in the state dynamics

$$\begin{aligned} x_{t+1} &= a(x_t - Q_K^{\Delta_t}(x_t) - U_N(e_t)) + d_t \\ &= a(e_t - U_N(e_t)) + d_t. \end{aligned} \tag{3}$$

In the case $N = 0$, this reduces to [10], [11] with

$$x_{t+1} = a(x_t - Q_K^{\Delta_t}(x_t)) + d_t. \tag{4}$$

We briefly motivate the given scheme. For the bin update rules we will describe below, the pair (x_t, Δ_t) forms a Markov chain. The update dynamics (4) with no fixed quantization ($N = 0$) ensure that the system is stochastically stable and ergodic, due to results of [11].

Our main contribution is analyzing the performance of the proposed two-stage scheme. Through an intricate stochastic stability analysis, we ultimately prove that the addition of a fixed quantization stage leads to optimal convergence of the second moment with near-optimality in the rate of convergence.

Finally, we describe the bin update dynamics. As in [10], [11], a simple *zooming* scheme is employed. Assuming that $K > |a|$ (which ensures stability), choose $\frac{|a|}{K} < \alpha < 1$, $\delta > 0$ and $L > 0$. The update is:

$$\Delta_{t+1} = \begin{cases} (|a| + \delta)\Delta_t, & \text{if } |x_t| > \frac{K}{2}\Delta_t \\ \alpha\Delta_t, & \text{if } |x_t| \leq \frac{K}{2}\Delta_t, \Delta_t \geq L \\ \Delta_t, & \text{if } |x_t| \leq \frac{K}{2}\Delta_t, \Delta_t < L. \end{cases} \tag{5}$$

The above rules imply that $\Delta_t \geq \alpha L$. We choose an arbitrary initial $\Delta_0 > 0$.

Then we consider the process $\{\phi_t\}_{t=0}^\infty = \{(x_t, \Delta_t)\}_{t=0}^\infty$ with the dynamics described above. This process is a Markov chain. The state space of this process highly depends on the following mild "countability condition" utilized in [10], [11].

Condition A. There exist relatively prime integers $j, k \geq 1$ such that $\alpha^j(|a|+\delta)^k = 1$. Equivalently, $\log_\alpha(|a|+\delta)$ is rational.

We restrict our analysis to the case where Condition A holds. We let the state space for Δ_t be

$$\Omega_\Delta := \left\{\alpha^j(|a|+\delta)^k\Delta_0 : j,k \in \mathbb{Z}_{\geq 0}\right\}.$$

The state space for the Markov chain $\{(x_t, \Delta_t)\}_{t=0}^\infty$ is then $\mathbb{R} \times \Omega_\Delta$.

C. Stochastic Stability

Suppose $\{\phi_t\}_{t=0}^\infty$ is a Markov chain with state space $\mathbb{X}$, where $\mathbb{X}$ is a complete separable metric space and its Borel sigma algebra is denoted $\mathcal{B}(\mathbb{X})$. The transition probability is denoted by P, so that for any $\phi \in \mathbb{X}$ and $A \in \mathcal{B}(\mathbb{X})$, the probability of moving in one step from state ϕ to the set A is given by $P(\phi_{t+1} \in A \mid \phi_t = \phi) = P(\phi, A)$. The n-step transitions are obtained in the usual way, $P(\phi_{t+n} \in A \mid \phi_t = \phi) = P^n(\phi, A)$ for any $n \geq 1$. The transition law acts on measurable functions $f : \mathbb{X} \to \mathbb{R}$ and measures μ on $\mathcal{B}(\mathbb{X})$ via

$$Pf(\phi) = \int_{\mathbb{X}} P(\phi, dy) f(y), \quad \text{for all } \phi \in \mathbb{X}$$

and

$$\mu P(A) = \int_{\mathbb{X}} \mu(d\phi) P(\phi, A), \quad \text{for all } A \in \mathcal{B}(\mathbb{X}).$$

A probability measure π on $\mathcal{B}(\mathbb{X})$ is called invariant if $\pi P = \pi$, that is:

$$\int_{\mathbb{X}} \pi(d\phi) P(\phi, A) = \pi(A), \quad \text{for all } A \in \mathcal{B}(\mathbb{X}).$$

For any initial probability measure ν on $\mathcal{B}(\mathbb{X})$ we can construct a stochastic process with transition law P and $\phi_0 \sim \nu$. We let P_ν denote the resulting probability measure on the sample space, with the usual convention that $\nu = \delta_\phi$ (i.e., $\nu(\{\phi\}) = 1$) when the initial state is $\phi \in \mathbb{X}$. When $\nu = \pi$ the resulting process is stationary.

There is at most one stationary solution under the following irreducibility assumption. For a set $A \in \mathcal{B}(\mathbb{X})$ we denote,

$$\tau_A := \min\{t \geq 1 : \phi_t \in A\}. \tag{6}$$

Definition 2.1. *Let φ denote a σ-finite measure on $\mathcal{B}(\mathbb{X})$.*

(i) The Markov chain is called *φ-irreducible* if for any $\phi \in \mathbb{X}$ and $B \in \mathcal{B}(\mathbb{X})$ satisfying $\varphi(B) > 0$ we have

$$P_\phi(\tau_B < \infty) > 0.$$

(ii) A φ-irreducible Markov chain is *Harris recurrent* if $P_\phi(\tau_B < \infty) = 1$ for any $\phi \in \mathbb{X}$ and any $B \in \mathcal{B}(\mathbb{X})$ satisfying $\varphi(B) > 0$. It is *positive Harris recurrent* if in addition there is an invariant probability measure π.

Notably, the positive Harris recurrence property leads to ergodicity of the closed-loop process: for every initial state and every $g \in L_1(\pi)$, the following holds almost surely: $\lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} g(\phi_k) = \int \pi(d\pi) g(\phi)$. This almost sure sample path convergence also holds in expectation under mild Lyapunov conditions [28] (which will be the case in our analysis).

III. ANALYSIS OF SCHEME

A. Supporting Lemma on Optimality at High–Rates

For the class of fixed quantizers U_N introduced in Section II-A, we present the following result bounding the expected MSE of a family of well-behaved random variables.

Lemma 3.1. *Suppose $\{X_N\}_{N=2}^\infty$ are random variables satisfying the following uniform moment bound:*

$$E\left[|X_N|^m\right] \leq B_m \quad \textit{for all } m = 1, 2, ... \tag{7}$$

where $\{B_m\}_{m=1}^\infty$ is independent of N. Let $\varepsilon > 0$ and set the bin size for U_N as $\Delta_{(N)} = AN^{-1+\frac{1}{2}\varepsilon}$, for fixed $A > 0$. Then we have

$$E\left[(X_N - U_N(X_N))^2\right] = \mathcal{O}\left(\frac{1}{N^{2-\varepsilon}}\right),$$

i.e., there exists a constant $C > 0$ which depends on ε with

$$E\left[(X_N - U_N(X_N))^2\right] \leq CN^{-2+\varepsilon}, \quad \textit{for all } N \geq 2.$$

Proof: For brevity, denote $Y_N = X_N - U_N(X_N)$. First note that when $|X_N| \leq \frac{1}{2}N\Delta_{(N)}$, we have $|Y_N| \leq \frac{1}{2}\Delta_{(N)}$. Therefore,

$$E\left[Y_N^2 \mathbf{1}_{\{|X_N| \leq \frac{1}{2}N\Delta_{(N)}\}}\right] \leq \tfrac{1}{4}\Delta_{(N)}^2 = \tfrac{1}{4}A^2N^{-2+\varepsilon}. \tag{8}$$

Next we will consider the region $|X_N| > \frac{1}{2}N\Delta_{(N)}$. Here, note that $Y_N = X_N$ since $U_N(X_N) = 0$. Therefore,

$$E\left[Y_N^2 \mathbf{1}_{\{|X_N| > \frac{1}{2}N\Delta_{(N)}\}}\right] = E\left[X_N^2 \mathbf{1}_{\{|X_N| > \frac{1}{2}N\Delta_{(N)}\}}\right].$$

We then apply Hölder's inequality with conjugates $\frac{2+\varepsilon}{\varepsilon}$ and $\frac{2+\varepsilon}{2}$:

$$\begin{aligned} &E\left[X_N^2 \mathbf{1}_{\{|X_N| > \frac{1}{2}N\Delta_{(N)}\}}\right] \\ &\quad \leq E\left[|X_N|^{2+\frac{4}{\varepsilon}}\right]^{\frac{\varepsilon}{2+\varepsilon}} E\left[\left(\mathbf{1}_{\{|X_N| > \frac{1}{2}N\Delta_{(N)}\}}\right)^{1+\frac{1}{2}\varepsilon}\right]^{\frac{2}{2+\varepsilon}} \\ &\quad = E\left[|X_N|^{2+\frac{4}{\varepsilon}}\right]^{\frac{\varepsilon}{2+\varepsilon}} P\left(|X_N| > \tfrac{1}{2}N\Delta_{(N)}\right)^{\frac{2}{2+\varepsilon}}. \end{aligned} \tag{9}$$

Now, choose an integer $m \geq 2 + \frac{4}{\varepsilon}$ such that $-\frac{m\varepsilon}{2+\varepsilon} \leq -2$. First, note that by Jensen's inequality and the uniform moment bound (7),

$$E\left[|X_N|^{2+\frac{4}{\varepsilon}}\right]^{\frac{\varepsilon}{2+\varepsilon}} \leq E\left[|X_N|^m\right]^{\frac{1}{m}\left(2+\frac{4}{\varepsilon}\right)\frac{\varepsilon}{2+\varepsilon}} \leq (B_m)^{\frac{2}{m}}.$$

Secondly, note that by Markov's inequality and the uniform moment bound, for any $u > 0$ we have

$$P(|X_N| > u) \leq \frac{E\left[|X_N|^m\right]}{u^m} \leq B_m u^{-m}.$$

Therefore,

$$P\left(|X_N| > \tfrac{1}{2}N\Delta_{(N)}\right)^{\frac{2}{2+\varepsilon}} \leq (B_m)^{\frac{2}{2+\varepsilon}} \left(\tfrac{1}{2}N\Delta_{(N)}\right)^{-\frac{2m}{2+\varepsilon}}.$$

We conclude that the product in (9) is upper bounded by

$$\begin{aligned}
&(B_m)^{\frac{2}{m}} (B_m)^{\frac{2}{2+\varepsilon}} \left(\tfrac{1}{2} N \Delta_{(N)}\right)^{-\frac{2m}{2+\varepsilon}} \\
&= (B_m)^{\frac{2}{m}+\frac{2}{2+\varepsilon}} \left(\tfrac{1}{2} A N^{\frac{1}{2}\varepsilon}\right)^{-\frac{2m}{2+\varepsilon}} \\
&= (B_m)^{\frac{2}{m}+\frac{2}{2+\varepsilon}} \left(\tfrac{1}{2} A\right)^{-\frac{2m}{2+\varepsilon}} \cdot N^{-\frac{m\varepsilon}{2+\varepsilon}} \\
&\leq (B_m)^{\frac{2}{m}+\frac{2}{2+\varepsilon}} \left(\tfrac{1}{2} A\right)^{-\frac{2m}{2+\varepsilon}} \cdot N^{-2} \\
&\leq (B_m)^{\frac{2}{m}+\frac{2}{2+\varepsilon}} \left(\tfrac{1}{2} A\right)^{-\frac{2m}{2+\varepsilon}} \cdot N^{-2+\varepsilon}. \qquad (10)
\end{aligned}$$

Combining (8) and (10) gives the upper bound $\mathcal{O}\left(N^{-2+\varepsilon}\right)$, completing the proof. □

The above lemma will allow us, via the second part of our code, to bound the rate at which the MSE for a random variable decreases to zero as we increase the number of quantization bins N.

B. System Stability and Rate Optimality

In this section, we characterize the system limiting second moment in terms of the data rate and establish key stability properties of the system. Recall that C is the data rate of our noiseless channel.

Consider the limiting second moment $\lim_{t\to\infty} E\left[x_t^2\right]$. It is obvious from the state update equation (3) that $\lim_{t\to\infty} E\left[x_t^2\right] \geq \sigma^2$, so we wish to derive bounds for the "rate" at which $\lim_{t\to\infty} E\left[x_t^2\right] - \sigma^2 \to 0$ as $C \to \infty$. To make this precise, we will say that the limiting second moment converges (to σ^2) at a rate $r > 0$ if

$$\limsup_{C\to\infty} 2^{rC} \left(\lim_{t\to\infty} E\left[x_t^2\right] - \sigma^2\right) < \infty.$$

The following result implies that regardless of the choice of encoding/control scheme, any such rate satisfies $r \leq 2$.

Lemma 3.2. *The following lower bound on the limiting second moment holds:*

$$\lim_{t\to\infty} E\left[x_t^2\right] - \sigma^2 \geq \frac{a^2\sigma^2}{2^{2C} - a^2}.$$

Proof: This result follows directly from [29, Theorem 11.3.2] (via entropy-power inequality) in which the system (1) is considered (with $b = 1$) for control across a general memoryless channel of capacity C. The theorem provides a lower bound on C in terms of the limiting second moment, which when applied to our system, provides a lower bound on the invariant second moment. □

The remaining contributions of this section are as follows. First, we establish key stability results (positive Harris recurrence and ergodicity) of our coding and control scheme. Secondly, we show that for any $\varepsilon > 0$ an appropriate choice of $\Delta_{(N)}$ ensures that the limiting second moment converges at a rate $r \geq 2-\varepsilon$. In this sense and in view of the previous lemma, we provide a scheme which is arbitrarily close to order-optimal in the data rate C among all possible schemes.

As stated earlier, we will assume that Condition A holds. Consider the case where $N = 0$. By [11, Theorem 3.1], if $K > |a|$, then $\{(x_t, \Delta_t)\}_{t=0}^{\infty}$ is a positive Harris recurrent Markov chain with unique invariant measure π. We offer the following extension:

Theorem 3.3. *For all $N \geq 0$, $\{(x_t, \Delta_t)\}_{t=0}^{\infty}$ is positive Harris recurrent.*

Sketch of Proof: The case $N \geq 2$ is different from the case of no fixed quantization only in that the state update dynamics are decreased deterministically. This allows us to extend the proof program due to [11] for positive Harris recurrence for the case $N = 0$ with only minor modifications to some of the intermediate expressions. □

Now suppose that the system is positive Harris recurrent with fixed parameters $(a, \sigma^2, K, L, \alpha, \delta)$ and given $N \geq 2$ (this is possible so long as $K > |a|$). Let π_N be the resulting invariant measure depending on N, and let $(x_N^*, \Delta_N^*) \sim \pi_N$. Since the adaptive quantization error e_t is a deterministic function of (x_t, Δ_t), this invariant measure will also induce a stationary distribution for e_t, which we denote by $e_N \sim \pi_N^{\text{err}}$.

Theorem 3.4. *For all $N \geq 2$, for every initial condition,*

$$\lim_{t\to\infty} E[x_t^2] = E[(x_N^*)^2] = \int \pi_N(dx, d\Delta) x^2.$$

Proof: This result follows from [11, Theorem 2.1(iii)] and the drift criteria we establish in Section III-C. □

To show that we can achieve r arbitrarily close to 2, we will apply Lemma 3.1 to the stationary adaptive error e_N, so it will be necessary to show that e_N admits uniformly bounded moments:

Lemma 3.5. *The stationary adaptive error $\{e_N\}_{N=2}^{\infty}$ has uniformly bounded moments, i.e.,*

$$E\left[|e_N|^m\right] \leq B_m \quad \text{for all } m = 1, 2, \ldots$$

for some $\{B_m\}_{m=1}^{\infty}$ independent of N.

Remark. While this result is not surprising, the proof program is quite involved. We discuss this in further detail in Section III-C.

Finally, it follows from Lemma 3.1 that we can achieve a rate $r \geq 2-\varepsilon$, for any $\varepsilon > 0$.

Theorem 3.6. *Choose any $\varepsilon > 0$ and set the bin size for U_N as $\Delta_{(N)} = AN^{-1+\frac{1}{2}\varepsilon}$, for fixed $A > 0$. Then we have*

$$E\left[(x_N^*)^2\right] - \sigma^2 = \mathcal{O}\left(\frac{1}{N^{2-\varepsilon}}\right), \qquad (11)$$

i.e., there exists $C > 0$ which depends on ε, such that

$$E\left[(x_N^*)^2\right] - \sigma^2 \leq C N^{-2+\varepsilon}, \quad \text{for all } N \geq 2.$$

Remark. This implies a rate of convergence $r \geq 2-\varepsilon$ since C and N are related by $2^C = (N+1)(K+1)$.

Proof: Let $(x_N^*, \Delta_N^*) \sim \pi_N$ and let $e_N = x_N^* - Q_K^{\Delta_N^*}(x_N^*)$ so that $e_N \sim \pi_N^{\text{err}}$. Now let x' be the one-step pushforward of x_N^* using e_N, that is

$$x' = a(e_N - U_N(e_N)) + Z, \qquad (12)$$

where $Z \sim \mathcal{N}(0, \sigma^2)$ is independent of e_N. Since we started under the invariant measure, x' is distributed identically to x_N^*. Furthermore, since $E\left[e_N^2\right]$ is finite by Lemma 3.5, it follows from (12) that $E\left[(x_N^*)^2\right] = E\left[(x')^2\right]$ is finite. For brevity, let $q_N = e_N - U_N(e_N)$, then by total expectation:

$$\begin{aligned} E\left[(x_N^*)^2\right] &= E\left[(x')^2\right] = E\left[E\left[(x')^2 \mid e_N\right]\right] \\ &= E\left[E\left[a^2 q_N^2 + 2aq_N Z + Z^2 \mid e_N\right]\right] \\ &= a^2 E\left[q_N^2\right] + 2aE\left[q_N\right] E\left[Z\right] + E\left[Z^2\right] \\ &= a^2 E\left[(e_N - U_N(e_N))^2\right] + \sigma^2. \end{aligned}$$

Therefore, $E\left[(x_N^*)^2\right] - \sigma^2 = a^2 E\left[(e_N - U_N(e_N))^2\right]$. Since e_N has uniformly bounded moments, Lemma 3.1 implies that our choice of $\Delta_{(N)}$ causes this MSE to be $\mathcal{O}\left(\frac{1}{N^{2-\varepsilon}}\right)$. The result follows since the optimality gap is linearly related to this MSE. □

C. Proof Program for Uniformly Bounded Moments

Here we give a brief overview of the proof program for Lemma 3.5. First, we remark that if the invariant state x_N^* admits moments uniformly bounded in N, so too does e_N and Lemma 3.5 follows. This follows since for $e_N = x_N^* - Q_K^{\Delta_N^*}(x_N^*)$ we have $|e_N| \leq |x_N^*|$.

To show that the state admits uniformly bounded moments, we introduce a set of random-time Lyapunov drift criteria and show that if these criteria are satisfied, we can upper bound the expectation of functions under invariant measure. As in Section II-C, we consider a quite general Markov chain $\{\phi_t\}_{t=0}^{\infty}$ with state space $\mathbb{X}$. First, suppose that $\{\mathcal{T}_i : i \in \mathbb{N}\}$ is a sequence of strictly increasing stopping times with $\mathcal{T}_0 = 0$. Then for a measurable function $V : \mathbb{X} \to (0, \infty)$, measurable functions $f, d : \mathbb{X} \to [0, \infty)$, a constant b and a set $C \subseteq \mathcal{B}(\mathbb{X})$, we say that $\{\phi_t\}_{t=0}^{\infty}$ satisfies the random-time Lyapunov drift criteria if for all $z = 0, 1, 2, \ldots$

$$E\left[V(\phi_{\mathcal{T}_{z+1}}) \mid \mathcal{F}_{\mathcal{T}_z}\right] \leq V(\mathcal{T}_z) - d(\phi_{\mathcal{T}_z}) + b\mathbf{1}_{\{\phi_{\mathcal{T}_z} \in C\}},$$

and

$$E\left[\sum_{t=\mathcal{T}_z}^{\mathcal{T}_{z+1}-1} f(\phi_t) \;\Big|\; \mathcal{F}_{\mathcal{T}_z}\right] \leq d(\phi_{\mathcal{T}_z}). \tag{13}$$

These drift criteria can be used to establish important stability properties, in combination with irreducibility and other conditions (see [11, Theorem 2.1]).

Finally, we apply this to our system which we know to be positive Harris recurrent. The stopping times we choose are those in which the system (x_t, Δ_t) is "in-view" of the adaptive quantizer, i.e.,

$$\mathcal{T}_{k+1} = \min\left\{t > \mathcal{T}_k : |x_t| \leq \tfrac{K}{2}\Delta_t\right\}.$$

Let $m \geq 1$ be arbitrary and choose arbitrary $\gamma \in (0, 1-\alpha^m)$. Let $\beta > 0$ and $D > 0$, then set

$$\begin{aligned} &V(x, \Delta) = \Delta^m, \qquad d(x, \Delta) = \gamma\Delta^m, \\ &f(x, \Delta) = \gamma\beta\left(\tfrac{2}{K}\right)^m |x|^m, \qquad C = \{(x, \Delta) : \Delta \leq D\}. \end{aligned}$$

We have the following key supporting result (for a proof, please see [30]):

Proposition 3.7. *There exists choice of β sufficiently small, D sufficiently large, and b sufficiently large, all independent of N, so that the system $\{(x_t, \Delta_t)\}_{t=0}^{\infty}$ satisfies the random-time Lyapunov drift criteria* (13) *for the above choice of (V, d, f, b, C). It follows that*

$$E\left[|x_N^*|^m\right] \leq \tfrac{b}{\gamma\beta}\left(\tfrac{K}{2}\right)^m =: B_m$$

and so the state admits uniformly bounded moments. By our initial discussion, so too does the adaptive error e_N.

This completes the proof sketch for Lemma 3.5.

IV. A SIMULATION

Let us note that a key issue in a numerical simulation here is that for ε too close to 0, it is not feasible to effectively demonstrate the rate of convergence. To be precise, suppose $N \leq 2^{2/\varepsilon}$. Then the quantizer support size satisfies $\frac{1}{2}N\Delta_{(N)} \leq A$. Therefore, all schemes with less than $2^{2/\varepsilon}$ bins will fail to quantize the region $(-\infty, A) \cup (A, \infty)$. As a direct result of this, the overload distortion will not go to zero as N increases up to this point. This is especially problematic for small ε. As an example, suppose $\varepsilon = 0.001$. We would then require our fixed quantizer to have $N \gg 2^{2000}$ bins in order to demonstrate the convergence rate $\mathcal{O}\left(N^{-1.999}\right)$. It is infeasible on typical computing software (e.g. MATLAB) to simulate a scheme with this many bins. In view of this, we now proceed with an example simulation. To ensure that the convergence is observable, we modestly choose $\varepsilon = 1$ so that the rate of convergence is at worst $\mathcal{O}\left(1/N\right)$. The system parameters are $a = 2$ and $\sigma^2 = 1$. $K = 16$ bins are used to stabilize the system, with zooming parameters $\alpha = 0.5$, $\delta = 0.2$, and $L = 1$. We start the simulation from $x_0 = 0$, $\Delta_0 = L$. With the parameters above, the system was run for all $N \in \{10, 12, \ldots\ 100\}$ and the average second moment was recorded. The second moment achieved in each trial for N is shown below. As the convergence is at worst $\mathcal{O}\left(1/N\right)$, we include an estimate of this order for the tail of the data.

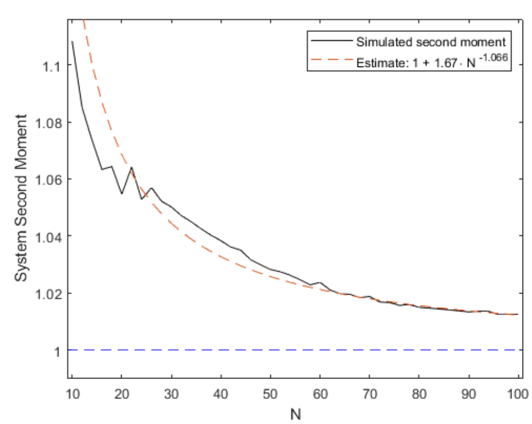

Fig. 3. Convergence to the optimum $\sigma^2 = 1$. The order of convergence is approximately $\mathcal{O}\left(N^{-1.066}\right)$ for the tail end of the data, overlaid in red.

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