Integration of Control and Information Theory: a 20 year personal account

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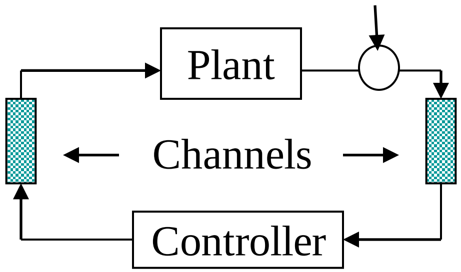
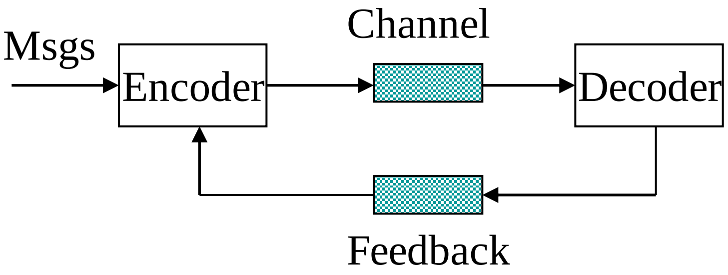
Thanks to my students, collaborators and sponsors

Intersection of Control and Information Theories



- Performance limitations by the communication channel?
- How to encode and decode for feedback?
- Do separation principles hold?
- Information theory for control systems? (Causality)

Mitter, Tatikonda, Sahai, Brockett, Liberzon, Varaiya, Basar, Yuksel, Baillieul, Nair, Evans, Savkin,Mateev, Sinopoli, Franceschetti, Martins, Dahleh, Gupta, Schienato,…



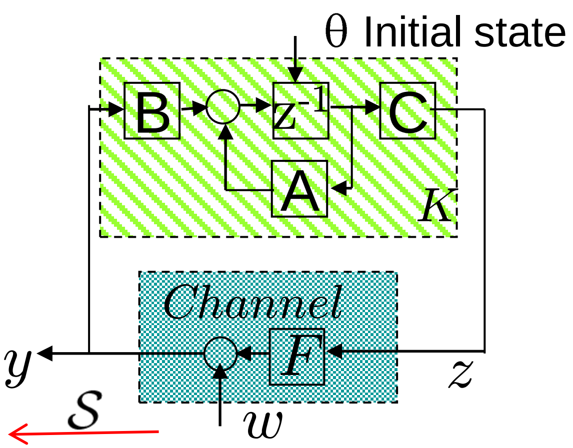
- Feedback is available in large networks.
- Performance improvement?
- Simpler encoders and decoders
- Control theory tools for feedback communication systems?

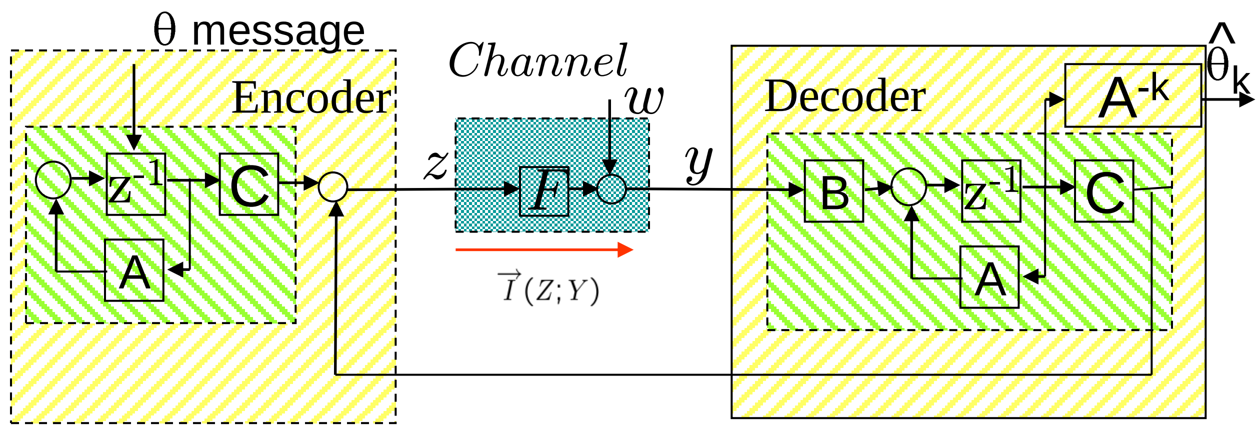
Shannon, Cover, Pombra, Kailath, Schalkwijk, Butman, Ozarow, Kramer, Massey, Tatikonda, Mitter, Kim …

Bode meets Shannon: Stabilization=Communication

[Elia TAC04]

Feedback Control System ⟺ Feedback Communication System





Stabilization
with A unstable

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \ln|\mathcal{S}(e^{j2\pi\theta})|d\theta = \sum_{i=1}^{m} \ln|\lambda_i(A_u)| = \overrightarrow{I}(Z;Y) = Rate$$

Bode Integral Formula = Directed Information = Degree of Instability

- Can use controller design tools to obtain feedback communication schemes
- Understand limitations of feedback in Information theory terms and vice-versa

$$\overrightarrow{I}(Z;Y) = \lim_{T\to\infty} \frac{1}{T} I(Z^T \to Y^T) = \sum_{t=0}^{T} I(Z^t; Y_t|Y^{t-1}) \text{ (Massey)}$$

Control-oriented Communication Schemes

- Shown agreement of fundamental limitations control communication and estimation. [Elia TAC04, Liu, Elia CIS 2014]
- Control inspired schemes are better then or equal to existing schemes
  - List of Applications: AWGN, Nth-order AR Gaussian Channel, Nth-order ISI Channel, MAC, BC, [Ardestanizadeh, Minero, Franceschetti 2012], Interference Channel, Fading channels with CSI, Dirty paper with feedback, [Liu, Elia CIS05], Markov channels with CSI, [Liu, Elia, Tatikonda, IT15]
- Control approach instrumental to the computation of **feedback capacity** of Stationary Gaussian channels. [Li and Elia, Alllerton 2015]
- Characterization of noisy feedback capacity and bounds. [Li, Elia ISIT 2011]
- Main implications for control systems
  - Gaussian channels are the least constraining channels
  - No need for encoders and decoders→ optimal communication at the physical layer.
  - Fading channels are more limiting for feedback systems (later)

Joint work of Abhishek Rawat [Rawat, Elia ITW 2020]

FEEDBACK CAPACITY OF ISI CHANNEL WITH COLORED NOISE

Selected works on Gaussian feedback capacity

- [Cover Pombra 1989] considered the time-varying additive Gaussian noise channel with feedback and characterized its capacity.
- [Kim 2010] considered the feedback capacity of stationary Gaussian channel with additive noise being colored.
- [Li, Elia 2018] provided algorithm to compute the capacity of Kim and extended the interpretation of feedback communication over stationary finite dimensional Gaussian channels as feedback control systems.
- [Gattami 2019] considered the state-space characterization of Kim and was able to formulate the problem in convex-optimization framework.

Motivation

- [Kim 2010] considered the scalar channel with colored noise having minimum-phase transfer function corresponding to its power spectral density.
- We generalized the approach in [Kim 2010] by considering the single user MIMO channel with ISI and additive colored noise. This allows us to study the channels with delays and non-minimum phase zeros which has not been done before.
- Frequency domain approach is insensitive to the non-minimum phase assumption while it is easy to get confused and obtain errorneus results when we apply the state-space approach in [Kim 2010],[Gattami 2019]

Channel Model

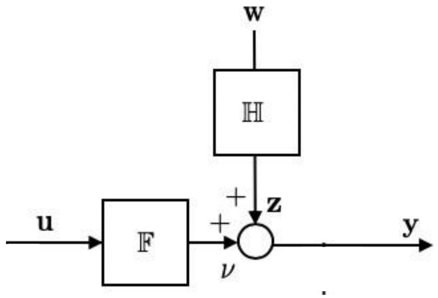


Figure: Block diagram of the Channel

- Single user MIMO channel with input $\mathbf{u} \in R^m$, multivariate white Gaussian noise $\mathbf{w} \in R^p$ and output $\mathbf{y} \in R^n$.
- Channel Model:

$$\mathbf{y} = \mathbb{F}\mathbf{u} + \mathbb{H}\mathbf{w}. \tag{1}$$

Channel Model

- Transfer function matrix $\mathbb{F}$ has minimal state-space representation:

$$x_{f(k+1)} = A_f x_{fk} + B_f \mathbf{u}_k,$$
$$\mathbf{y}_k = C_f x_{fk} + D_f \mathbf{u}_k + \mathbf{z}_k$$

- Transfer function matrix $\mathbb{H}$ has minimal state-space representation:

$$x_{h(k+1)} = A_h x_{hk} + B_h \mathbf{w}_k,$$
$$\mathbf{z}_k = C_h x_{hk} + D_h \mathbf{w}_k$$

- Assumptions
  - $\mathbb{F}$, $\mathbb{H}$ stable, $\mathbb{H}$ has no zeros on the unit circle
  - $\mathbb{S}(z) = \mathbb{H}(e^{j\theta})\mathbb{H}(e^{j\theta})^H > 0, \forall \theta \in [-\pi, \pi]$
  - Initial channel state not known to encoder and decoder

Definitions

- *Average Directed Information from input* $\mathbf{u}^N$ *to output* $\mathbf{y}^N$ *is given by:*

$$\overline{I(\mathbf{u}^N \rightarrow \mathbf{y}^N)} = \frac{1}{2N} \log \frac{det\Sigma_{\mathbf{y}^N}}{det\Sigma_{\mathbf{z}^N}}$$

- Limiting expression of DI *from input* $\mathbf{u}^N$ *to output* $\mathbf{y}^N$ *is*

$$I(\mathbf{u} \rightarrow \mathbf{y}) = \lim_{N\rightarrow\infty} \overline{I(\mathbf{u}^N \rightarrow \mathbf{y}^N)} = \lim_{N\rightarrow\infty} \frac{1}{2N} \log \frac{det\Sigma_{\mathbf{y}^N}}{det\Sigma_{\mathbf{z}^N}}$$

- *Limiting expression for average input power is given by*

$$\lim_{N\rightarrow\infty} \frac{1}{N} Tr\{E[\mathbf{u}^N \mathbf{u}^{NT}]\} \leq P$$

- *The limits may or may not exist*

Feedback Capacity

- For Gaussian Channels [Cover Pombra 1989]

$$C_{fb}(P) = \lim_{N\to\infty} \max_{\frac{1}{N}Tr\{E[\mathbf{u}^N \mathbf{u}^{NT}]\}\leq P} \frac{1}{2N} \log \frac{det\Sigma_{\mathbf{y}^N}}{det\Sigma_{\mathbf{z}^N}}$$

- Optimal input distribution is stationary for stable $\mathbb{H}$ [Kim 2010]

Frequency Domain Characterization of C_{fb}

$$
\begin{aligned}
&C_{fb}(\mathscr{P}) = \\
&\max_{\mathbb{Q},\mathbb{S}_v \succeq 0} \frac{1}{4\pi}\int_{-\pi}^{\pi} \log \frac{\det\{(I_n+\mathbb{F}\mathbb{Q})\mathbb{S}_z(I_n+\mathbb{F}\mathbb{Q})^H + \mathbb{F}\mathbb{S}_v\mathbb{F}^H\}}{\det \mathbb{S}_z} d\theta, \\
&\text{s.t.} \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{trace}\{\mathbb{Q}\mathbb{S}_z\mathbb{Q}^H + \mathbb{S}_v\} d\theta \leq \mathscr{P}, \\
&\qquad \mathbb{Q} \in \mathscr{R}\mathscr{H}_\infty^{m\times n}: \quad \text{strictly causal.}
\end{aligned}
$$

where, $\mathbb{S}_z(e^{i\theta}) = \mathbb{H}(e^{i\theta})\mathbb{H}(e^{i\theta})^H \succ 0, \quad \forall\theta \in [-\pi,\pi).$

- Extend the stationarity proof of [Kim 2010]
- Extend [Li Elia 2018]

Sketch of Proof

- Form of optimal input $\mathbf{u}^N = Q_N \mathbf{z}^N + \mathbf{v}^N$, where Q_N is strictly lower block triangular matrix.
- Output $\mathbf{y}^N = (I_N + F_N Q_N)\mathbf{z}^N + F_N \mathbf{v}^N$, F_N being the lower block triangular matrix.
- This enables us to formulate the average Directed Information in block form
- We follow the steps of Theorem 3.2 in [Kim 2010] to prove the stationarity of optimal input distributions, which enables us to use Szego limit theorems for the block Toeplitz matrices and obtain the expression for $I(\mathbf{u} \to \mathbf{y})$ in frequency domain.

A Comment on Special Case $\mathbb{F}$=I

$$
\begin{aligned}
& C_{fb}(\mathscr{P}) = \\
& \quad \max_{\mathbb{Q}, \mathbb{S}_v \succeq 0} \frac{1}{4\pi} \int_{-\pi}^{\pi} \log \frac{\det\{(I_n + \mathbb{Q})\mathbb{S}_z(I_n + \mathbb{Q})^H + \mathbb{S}_v\}}{\det \mathbb{S}_z} d\theta, \\
& \quad \text{s.t.} \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{trace}\{\mathbb{Q}\mathbb{S}_z\mathbb{Q}^H + \mathbb{S}_v\} d\theta \leq \mathscr{P}, \\
& \quad \quad \quad \mathbb{Q} \in \mathscr{R}\mathscr{H}_\infty^{m \times n} : \quad \text{strictly causal.}
\end{aligned}
\tag{1}
$$

where, $\mathbb{S}_z(e^{i\theta}) = \mathbb{H}(e^{i\theta})\mathbb{H}(e^{i\theta})^H \succ 0, \quad \forall \theta \in [-\pi, \pi).$

- Capacity only depends on the PSD of **z,** it does not depend on NMP zeros of $\mathbb{H}$
- The Capacity characterization of [Gattami 2019] is valid if $\mathbb{H}$, is MP.

Spectral Factorization

If (A_h, C_h) detectable, the channel can be alternatively written as:

$$\mathbf{y} = \mathbb{F}\mathbf{u} + \mathbb{H}_{mp}\tilde{\mathbf{w}}$$

where, $\mathbb{H}_{mp}$ is the outer function of $\mathbb{H} = \mathbb{H}_{mp}\mathbb{H}_i$ such that:

$$\mathbb{H}_{mp} = \left[\begin{array}{c|c} A_h & \Gamma \\ \hline C_h & I \end{array}\right], \quad \mathbb{H}_i = \left[\begin{array}{c|c} A_h - \Gamma C_h & B_h - \Gamma D_h \\ \hline C_h & D_h \end{array}\right].$$

where, $\Gamma = (A_h S C_h^T + B_h D_h^T)(C_h S C_h^T + D_h D_h^T)^{-1}$ is obtained from the Riccati Eq.

$$S = A_h S A_h^T + B_h B_h^T - \Gamma(C_h S C_h^T + D_h D_h^T)\Gamma^T \tag{1}$$

and $\tilde{\mathbf{w}} \sim \mathcal{N}(\mathbf{0}, \alpha)$, where $\alpha = C_h S C_h^T + D_h D_h^T$.

Modified Channel

$$\mathbf{y} = \mathbb{F}\mathbf{u} + \mathbb{H}_{mp}\tilde{\mathbf{w}}$$

$$\begin{bmatrix} s_{k+1} \\ \mathbf{y}_k \\ \mathbf{u}_k \end{bmatrix} = \left[\begin{array}{cc|cc} A_f & 0 & 0 & B_f \\ 0 & A_h & \Gamma & 0 \\ \hline C_f & C_h & I & D_f \\ 0 & 0 & 0 & I \end{array}\right] \begin{bmatrix} s_k \\ \tilde{\mathbf{w}}_k \\ \mathbf{u}_k \end{bmatrix}$$

$$\begin{aligned} s_{k+1} &= As_k + B_w\tilde{\mathbf{w}}_k + B\mathbf{u}_k, \\ \mathbf{y}_k &= Cs_k + D_w\tilde{\mathbf{w}}_k + D\mathbf{u}_k. \end{aligned}$$

Steps

- Entropy of **z**

$$h(\mathbf{z}) = \frac{1}{2}\log(2\pi e \det(C_h X C_h^T + D_h D_h^T))$$

X unique solution $X \succeq 0$ of Riccati Eq.

$$X = A_h X A_h^T + B_h B_h^T - (A_h X C_h^T + B_h D_h^T)(C_h X C_h^T + D_h D_h^T)^{-1}(A_h X C_h^T + B_h D_h^T)^T$$

- Optimal input for the equivalent channel is

$$\mathbf{u}_k = G(s_k - E[s_k|\mathbf{y}^{k-1}]) + \mathbf{v}_k$$

for some $G \in R^{m\ (sf+sh)}$.

Entropy of $\mathbf{y}$

- Assume that (A + BG, C + DG) is detectable, then

$$h(\mathbf{y}) = \frac{1}{2}\log(2\pi e \det \Sigma_y).$$

where

$$\Sigma_{\mathbf{y}} = (C+DG)\Sigma(C+DG)^T + DVD^T + \alpha.$$

and $\Sigma \geq 0$ unique solution of Riccati Eq.

$$\Sigma = (A+BG)\Sigma(A+BG)^T + BVB^T + B_w \alpha B_w^T - \Theta \Sigma_{\mathbf{y}} \Theta^T.$$

where $\Theta = \{(A+BG)\Sigma(C+DG)^T + BVD^T + B_w\alpha\}\Sigma_{\mathbf{y}}^{-1}.$

Steps are extension to our setup of similar steps in [Kim 2010] [Gattami 2019]

Non-Convex Characterization

$$
\begin{aligned}
C_{fb}(\mathscr{P}) = \max_{\Sigma \succeq 0, V \succeq 0, G} \quad & \frac{1}{2} \log\det \Sigma_{\mathbf{y}} - \frac{1}{2} \log \alpha, \\
s.t. \quad & \Sigma_{\mathbf{y}} = ((C+DG)\Sigma(C+DG)^T + DVD^T + \alpha, \\
& M_1 = (A+BG)\Sigma(A+BG)^T + BVB^T + B_w \alpha B_w^T, \\
& M_2 = (A+BG)\Sigma(C+DG)^T + BVD^T + B_w \alpha, \\
& \Sigma = M_1 - M_2 \Sigma_{\mathbf{y}}^{-1} M_2^T, \\
& \mathrm{trace}\{G\Sigma G^T + V\} = \mathscr{P}.
\end{aligned}
$$

Convex Optimization

$$
\begin{aligned}
C_{fb}(\mathscr{P}) = &\max_{\Sigma \succeq 0, K, P \succeq 0, V \succeq 0} \quad \frac{1}{2}\log\det\Sigma_{\mathbf{y}} - \frac{1}{2}\log\alpha. \\
&s.t \quad \Sigma_{\mathbf{y}} = C\Sigma C^T + DKC^T + CK^T D^T + DPD^T + \alpha, \\
&M_1 = A\Sigma A^T + BKA^T + AK^T B^T + BPB^T + B_w \alpha B_w^T \\
&M_2 = A\Sigma C^T + BKC^T + AK^T D^T + BPD^T + B_w \alpha, \\
&\begin{bmatrix} M_1 - \Sigma & M_2 \\ M_2^T & \Sigma_{\mathbf{y}} \end{bmatrix} \succeq 0, \\
&\begin{bmatrix} P - V & K \\ K^T & \Sigma \end{bmatrix} \succeq 0, \\
&\mathrm{trace}(P) = \mathscr{P}.
\end{aligned}
$$

[Rawat Elia ITW 2020]

So… what about feedback ?

- Recall Frequency characterization

$$
\begin{aligned}
C_{fb}(\mathscr{P}) = & \\
& \max_{\mathbb{Q},\mathbb{S}_v \succeq 0} \frac{1}{4\pi} \int_{-\pi}^{\pi} \log \frac{\det\{(I_n + \mathbb{Q})\mathbb{S}_z (I_n + \mathbb{Q})^H + \mathbb{S}_v\}}{\det \mathbb{S}_z} d\theta, \\
& \text{s.t.} \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{trace}\{\mathbb{Q}\mathbb{S}_z\mathbb{Q}^H + \mathbb{S}_v\} d\theta \leq \mathscr{P}, \\
& \qquad \mathbb{Q} \in \mathscr{R}\mathscr{H}_\infty^{m\times n} : \quad \text{strictly causal.}
\end{aligned}
$$

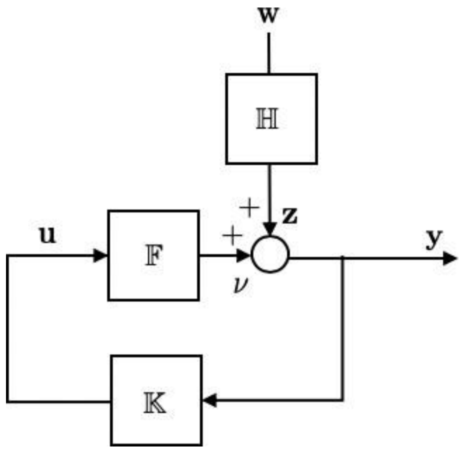
where, $\mathbb{S}_z(e^{i\theta}) = \mathbb{H}(e^{i\theta})\mathbb{H}(e^{i\theta})^H \succ 0, \quad \forall \theta \in [-\pi, \pi).$

- We are actually after $\mathbb{Q}$. SDP gives a LTI FD stable $\mathbb{Q}$
- All Stabilizing controllers of $\mathbb{F}$ are $\mathbb{K} = (I - \mathbb{Q}\mathbb{F})^{-1}$ for any $\mathbb{Q}$ stable
- When we find $\mathbb{Q}$, we find a stabilizing feedback controller for $\mathbb{F}, (\mathbb{H})$

[Elia 2004, Li, Elia 2018]

So… what about feedback ?

Once we find $\mathbb{Q}$, we find $\mathbb{K}$

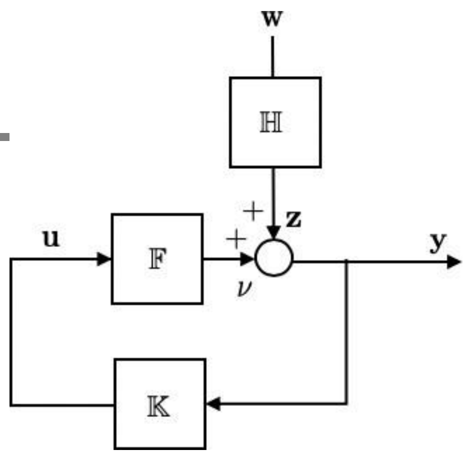


- We have an LTI stable feedback loop [Elia 2004]

$$C_{fb}(\mathscr{P}) = \log DI(\mathbb{K}) = \log \sum_{i=1}^{p} pk_i$$

- We have encoder and decoders [Liu Elia 2005, 2013]
- With double exponential probability of error

Example AR colored noise



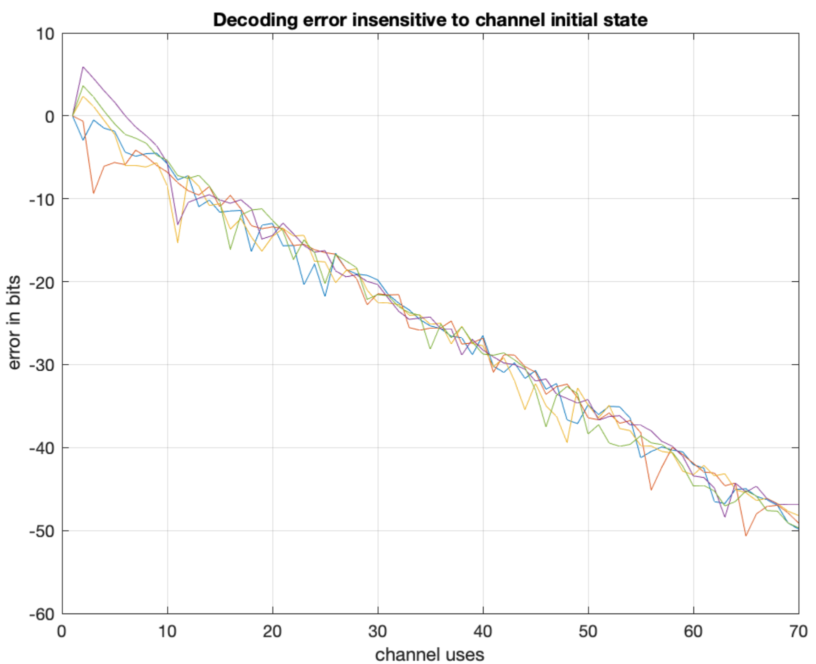
$$\mathbb{F} = 1; \qquad \mathbb{H} = \frac{z}{z+0.5}; \qquad \mathscr{P} = 1$$

$$C_{fb}(1) = 0.7167\ b.p.c.u; \qquad C(1) = 0.5964\ b.p.c.u.$$

$$\mathbb{Q}(z) = \frac{-0.79358(z+0.5)}{z(z-0.6083)}$$

$$\mathbb{K}(z) = \frac{-0.79358(z+0.5)}{(z+0.2414)(z-1.643)}$$

$$C_{fb} = 0.7167 = \log(1.643) \simeq \frac{50}{70}$$



Example MIMO

$$\mathbb{F} = \begin{bmatrix} \frac{z+2}{z+0.7} & \frac{z+1.5}{z+0.6} \\ \frac{z-1.6}{z+0.5} & \frac{z-3}{z+0.4} \end{bmatrix}, \mathbb{H} = \begin{bmatrix} \frac{z^2+2z+1.05}{z^2} & \frac{0.1(z+0.2)}{z} \\ \frac{0.05(z+0.3)}{z} & \frac{z^2+1.2z+1.06}{z^2} \end{bmatrix}$$

$$C_{fb}(10) \quad = \quad 5.2756 \quad b.p.c.u$$

$$C(10) \quad = \quad 4.8937 \quad b.p.c.u$$

Example MIMO

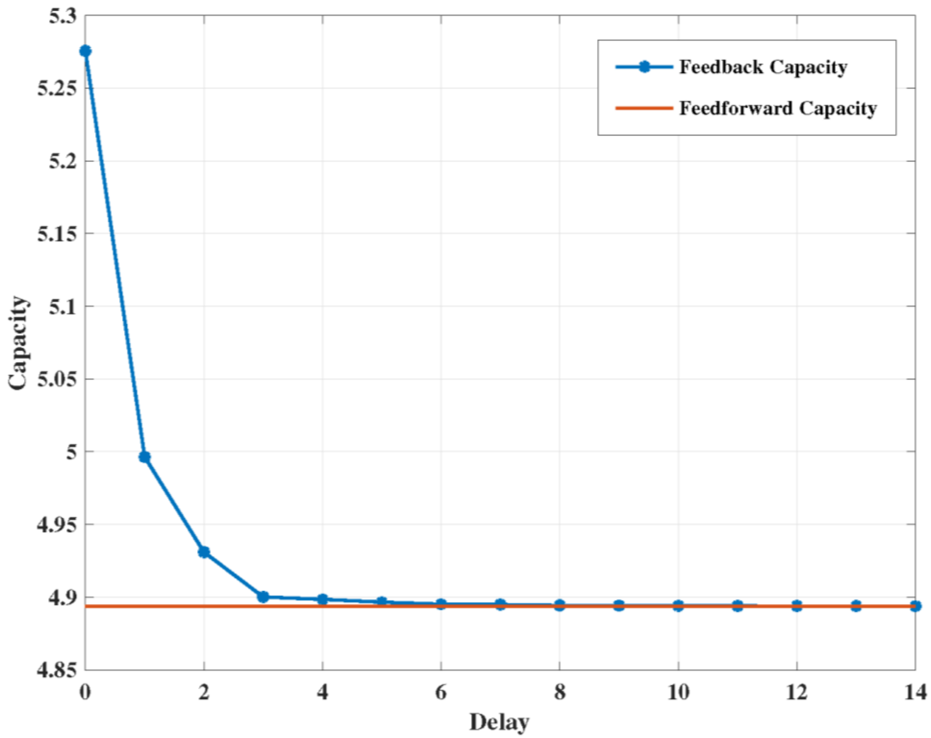
Effect of Delays

$$\mathbb{F} = \begin{bmatrix} \frac{z+2}{z+0.7} & \frac{z+1.5}{z+0.6} \\ \frac{z-1.6}{z+0.5} & \frac{z-3}{z+0.4} \end{bmatrix}, \mathbb{H} = \begin{bmatrix} \frac{z^2+2z+1.05}{z^2} & \frac{0.1(z+0.2)}{z} \\ \frac{0.05(z+0.3)}{z} & \frac{z^2+1.2z+1.06}{z^2} \end{bmatrix}$$

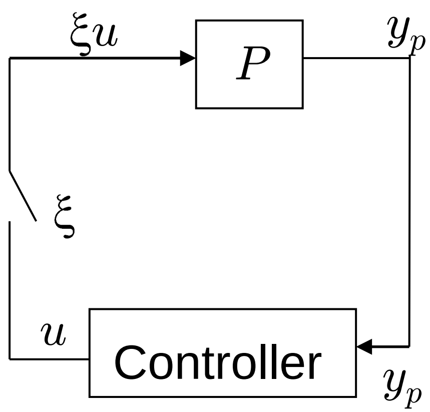
- Add delays to $\mathbb{F}$

$$C_{fb}(10) = 5.2756 \quad b.p.c.u.$$

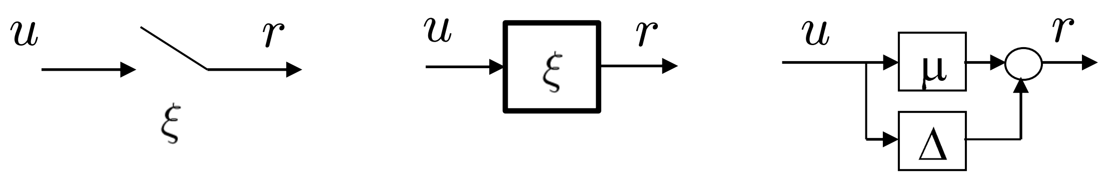
$$C(10) = 4.8937 \quad b.p.c.u.$$



Control over Fading Channels



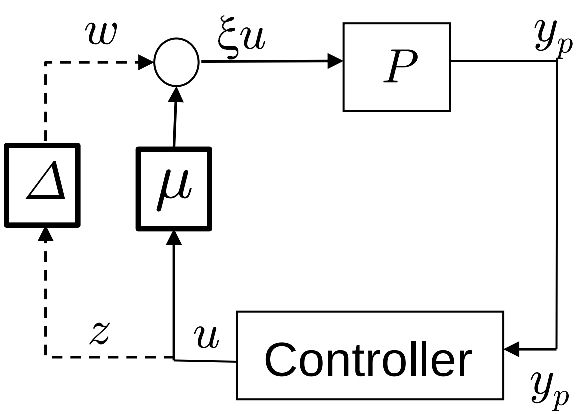
Stochastic uncertainty model of intermittent/erasure/fading link



$\xi(k)$ ~ Bernoulli IID mean = μ = (1-e), var = σ^2

$\xi(k) = \mu + \Delta(k)$; $\Delta(k)$ ~ mean = 0, var = σ^2

- Simple model of interaction between systems
- Model for packet loss in networks
- Special case of analog memory-less multiplicative channel
- Extends to Gaussian fading channels with memory

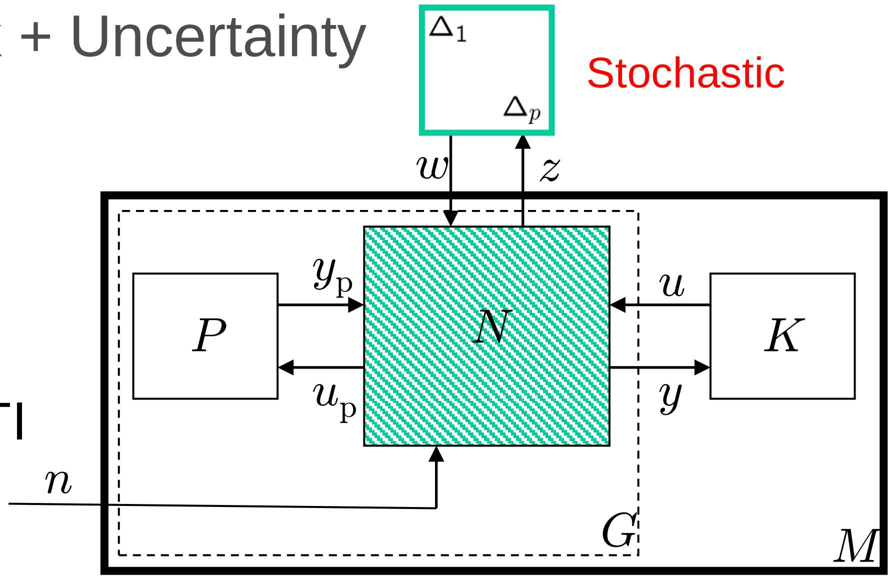


Focus on Mean Square Stability
Robustness to stochastic uncertainty

A Framework for Interconnected Systems

[Elia SCL05]

Fading Network = Mean Network + Uncertainty



- Uncertainty is Stochastic

  $\Delta_i(k)$ IID in k, Independent in i
  Zero Mean, var $= \sigma^2$

- Mean Network, N, deterministic LTI
- P and K LTI

M deterministic LTI

- Mean Square Stability (MSS)

System MS stable iff $\sigma^2 \rho(\hat{M}) < 1$

Separation result

$$\hat{M} = \begin{pmatrix} \|M_{11}\|_2^2 & \cdots & \|M_{1p}\|_2^2 \\ \vdots & \cdots & \vdots \\ \|M_{p1}\|_2^2 & \cdots & \|M_{pp}\|_2^2 \end{pmatrix}$$

- Robust control with stochastic uncertainty. $\rho(\ )$ = spectral radius

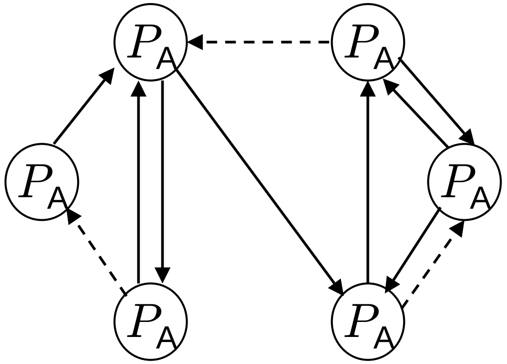
Based on ElGaoui 95, Ku Athans 77, Willems Blankenship 71, Kleinman 69 Wonham 67.

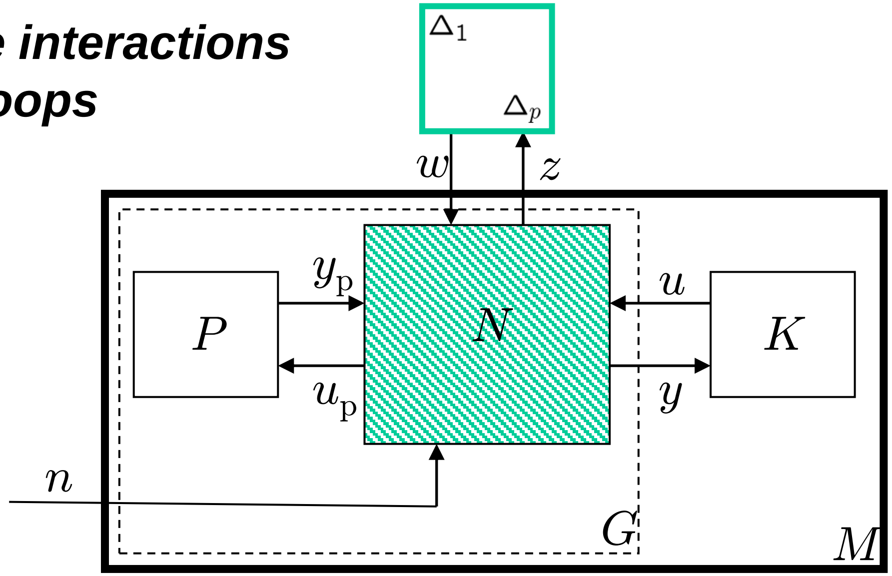
COMPLEX BEHAVIOR IN NETWORKED SYSTEMS

Communication Constraints

- How do communication channels or uncertain interactions affect networked systems?
- Concentrate on channel intermittency and additive noise

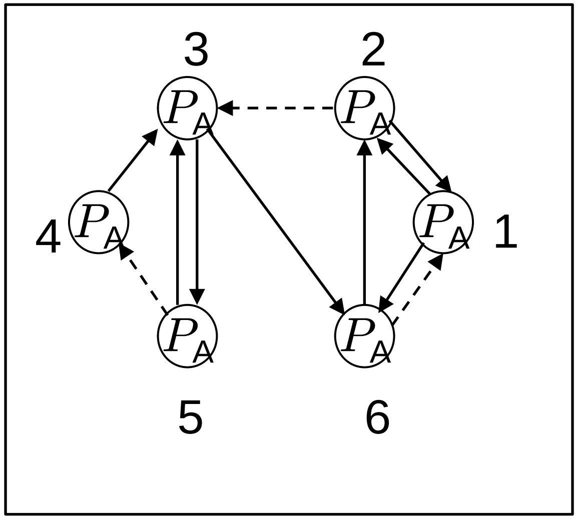
- ***Uncertainty in the interactions***
- ***Many feedback loops***





Focus on multi-agent systems with “simple” agents

Consensus: a Paradigm for Distributed Computation

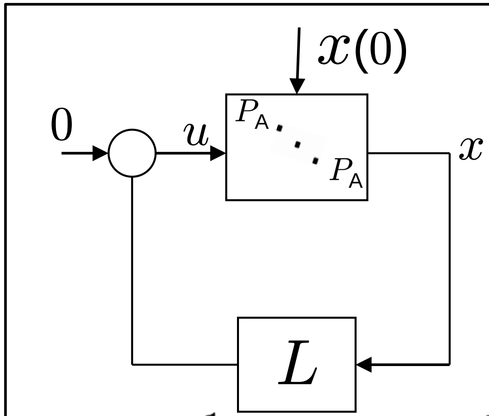


Discrete-time

$$x_i^+ = x_i + \beta \sum_{j \in N_i} (x_j - x_i)$$

Continuous-time

$$\dot{x}_i = \sum_{j \in N_i} (x_j - x_i)$$



$$P_A = \beta \frac{1}{z-1}, \text{ or } P_A = \frac{1}{s}$$

Each node use the relative error from its neighbors to update its own state.
The neighbors are determined by a graph: directed strongly connected, balanced
Property of graph Laplacian L**1**=**0.**

Under certain conditions $$\lim_{t \to \infty} x_i(t) = \frac{1}{n} \mathbf{1}^T x(0)$$

Tsitsiklis, Olfati-Saber,Scutari, Fax, Murray, Zampieri, Fagnani, Cortes, Pesenti, Moura, Kar, Giulietti, Ren, Beard, Papachristodoulou, Lee, Jadbabaie, Low,….

Limitations on Information Exchange

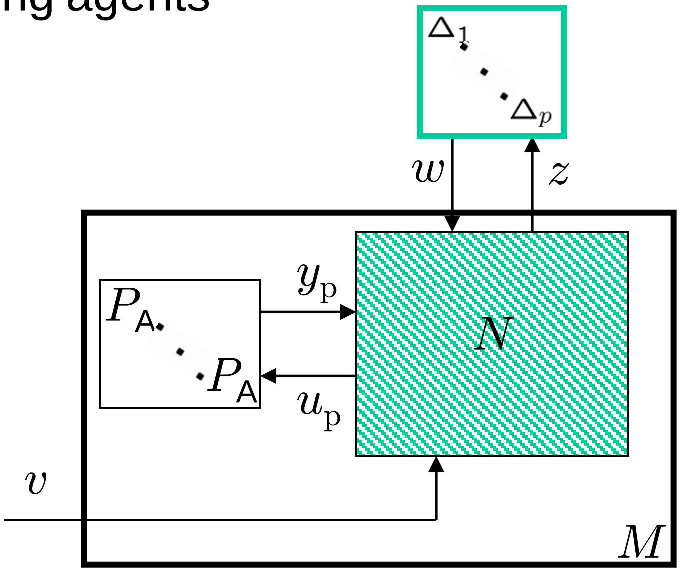
Averaging over unreliable channels + noise ?

$$x_i(k+1) = x_i(k) + \beta \sum_{j \in N_i} \xi_{ij}(k)[x_j(k-\tau_{ij}) - x_i(k)] + v_i(k)$$

Dropouts Delays Noise

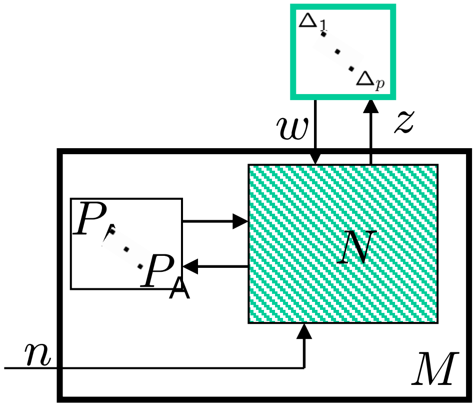
The model describes very simple-minded interacting agents
Simple model for natural behavior (flocking etc.)

- Apply the fading network framework
- M has structure

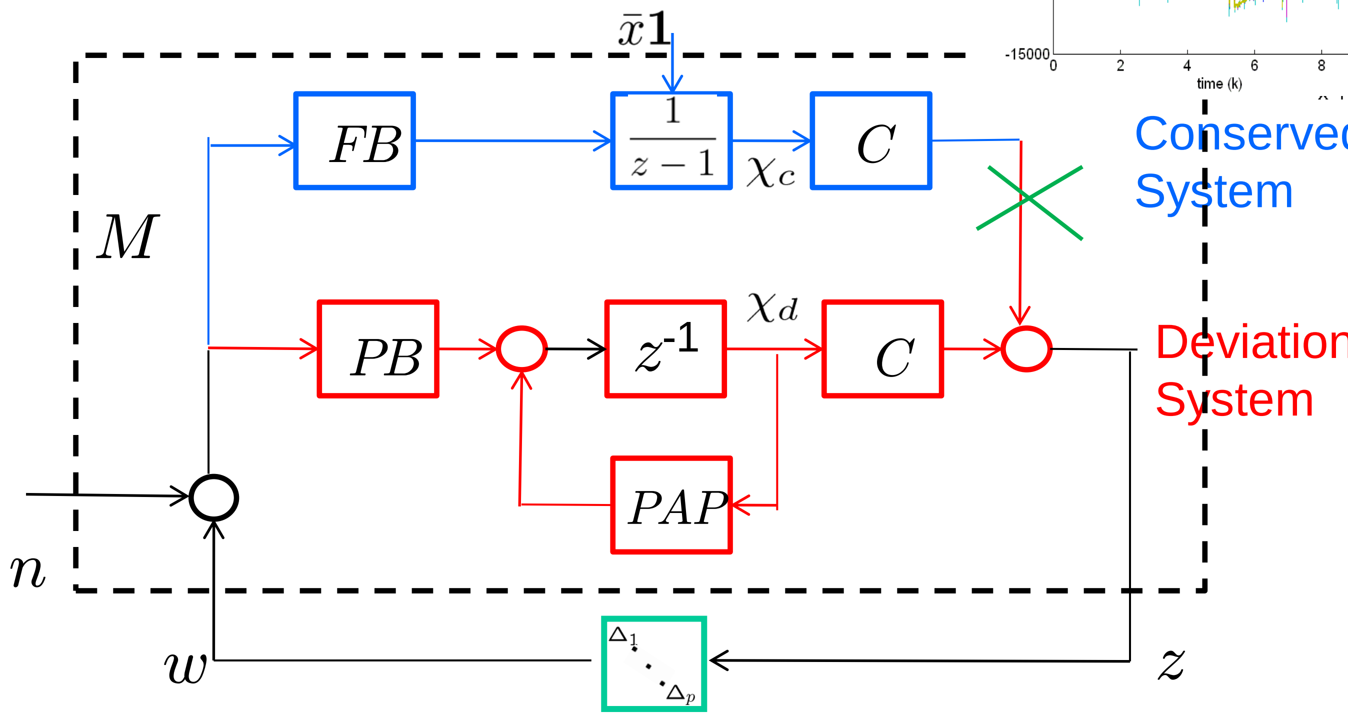


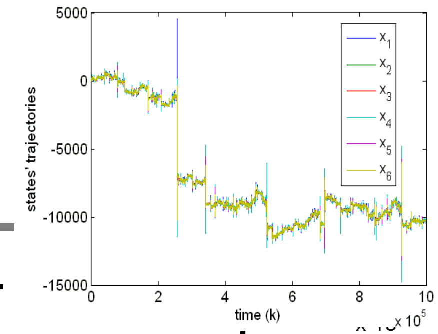
Assume $\mu_{ij} = \mu$ for simplicity

System Decomposition: Block Diagram



M has structure





Decomposition: Conserved + Deviation state $\chi = \chi_c + \chi_d$

$$\chi_c = \frac{1}{n}1^T\chi,$$

- Perfect links (n=0, Δ=0) → χ_c is equal to the average, $\bar{x}\mathbf{1}$
- Noisy links (Δ=0) → χ_c random walks
- Noise + fading → χ_c exhibits certain complex behavior if MSS is lost

Emergence of new collective complex behavior

[Wang Elia TAC12]

For directed IID switching and strongly connected mean graph, assume the deviation system converges to an invariant distribution. Then

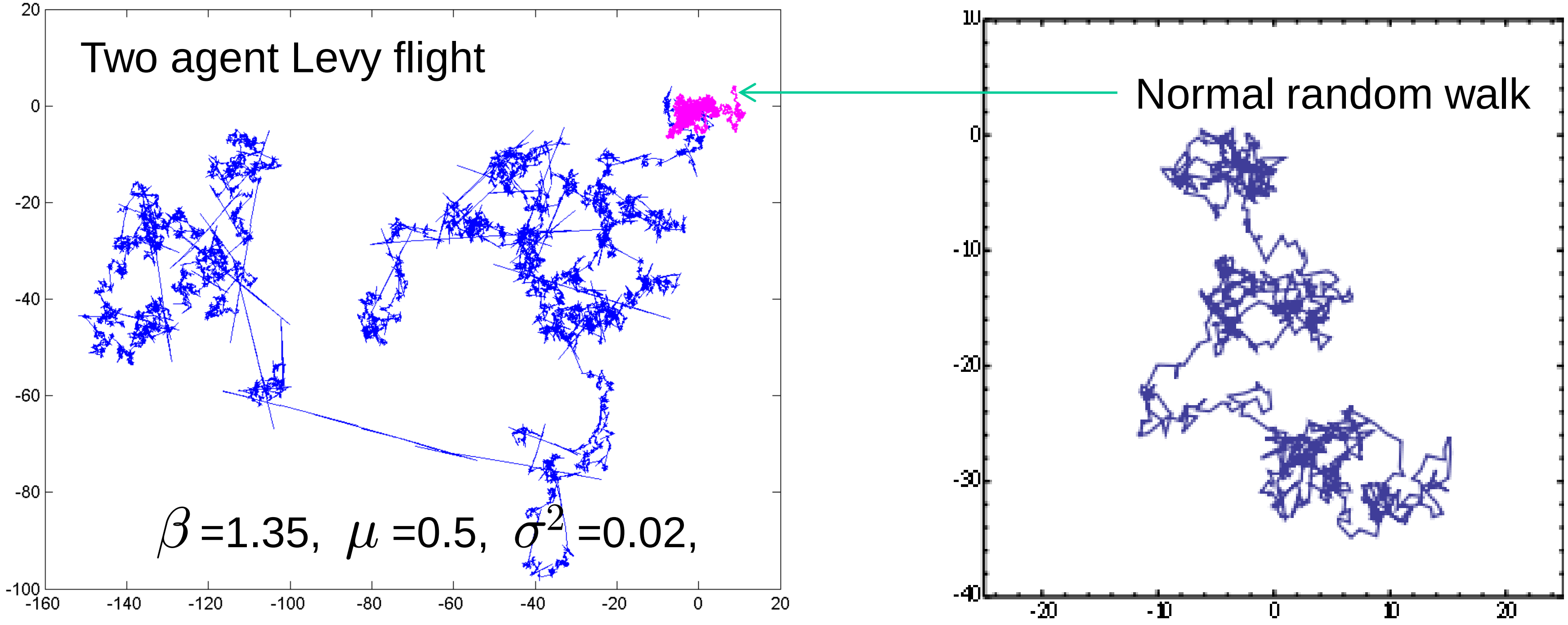
χ_c is a hyper-jump-diffusion $\lim_{k\to\infty}\{\chi_c(k)-\chi_c(k-1)\} \overset{dis}{=} R, \quad \mathbf{E}\{RR'\}=\infty$

χ_d is heavy tailed with unbounded second moment

$\Longleftrightarrow$ Deviation system is Mean Square unstable

- χ_c is an uncorrelated Levy flight, $\lim_{t\to\infty} t^{\alpha}\Pr(|R|>t)>0,\ 0<\alpha\leq 2$ for a two-node system (Kesten)
- Emergent complex behavior is global (collective)
- Long range impact of local criticality.

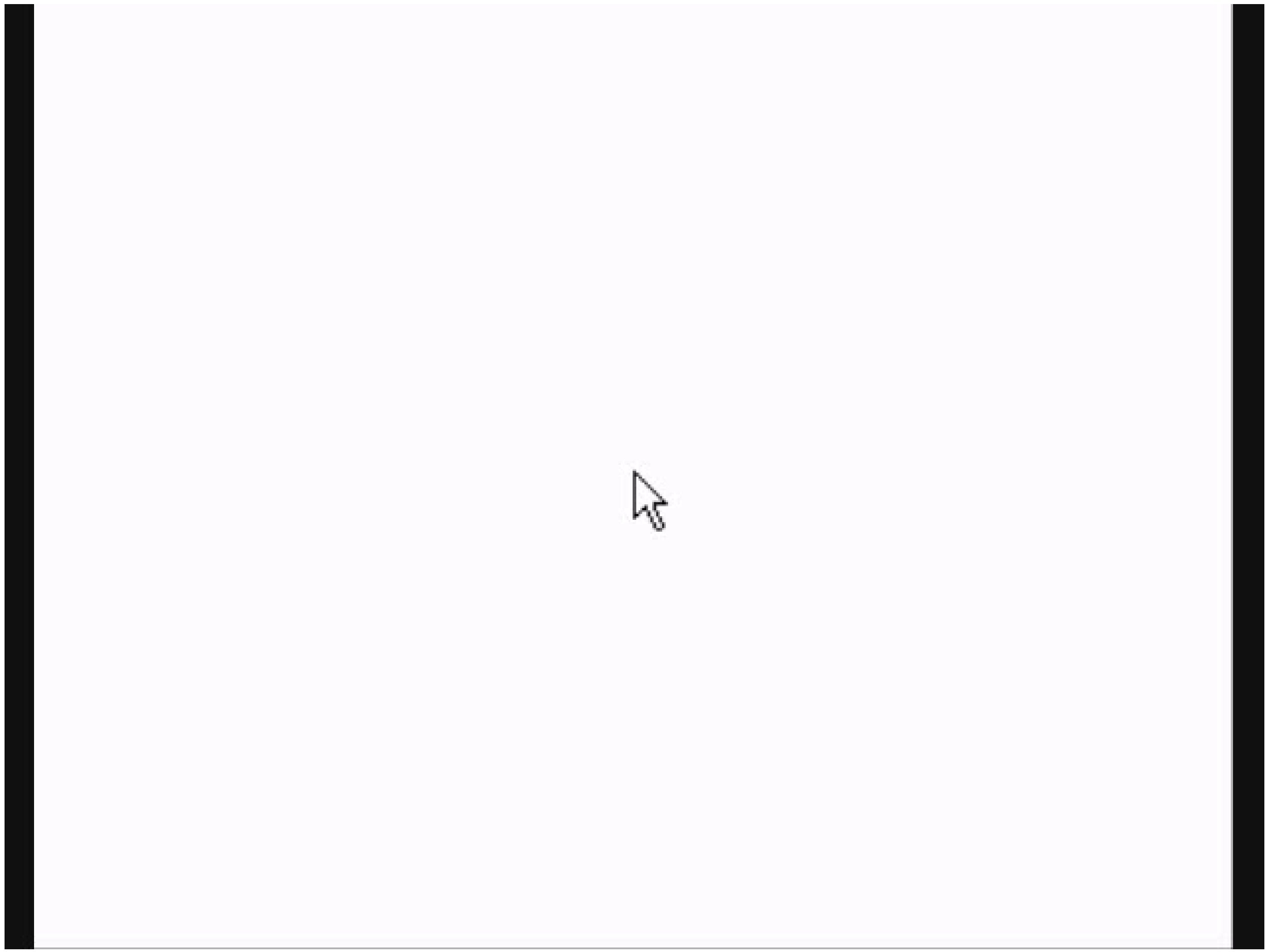
Levy flights vs. Normal Random Walk



- In the distribution of human travel [Brockmann]
- In economics and financial series [Mandelbrot, Sornette, Mantegna]
- In foraging search patterns of several species [Raynolds, Bartumeus]
- **Exploitation** cooperative searches and optimization?
- **Mitigation** strategies ?

MS Unstable Consensus no Noise

n=10
d=4
β =0.9
e=0.9
Noise var.=0
Delays 5



MS Unstable Consensus with Channel Noise

Emergence of complex behavior

Consensus system

- 10 nodes
- 4 neighbhds
- β =0.9
- e=0.9
- 5–step delay
- Noise var.1e-6

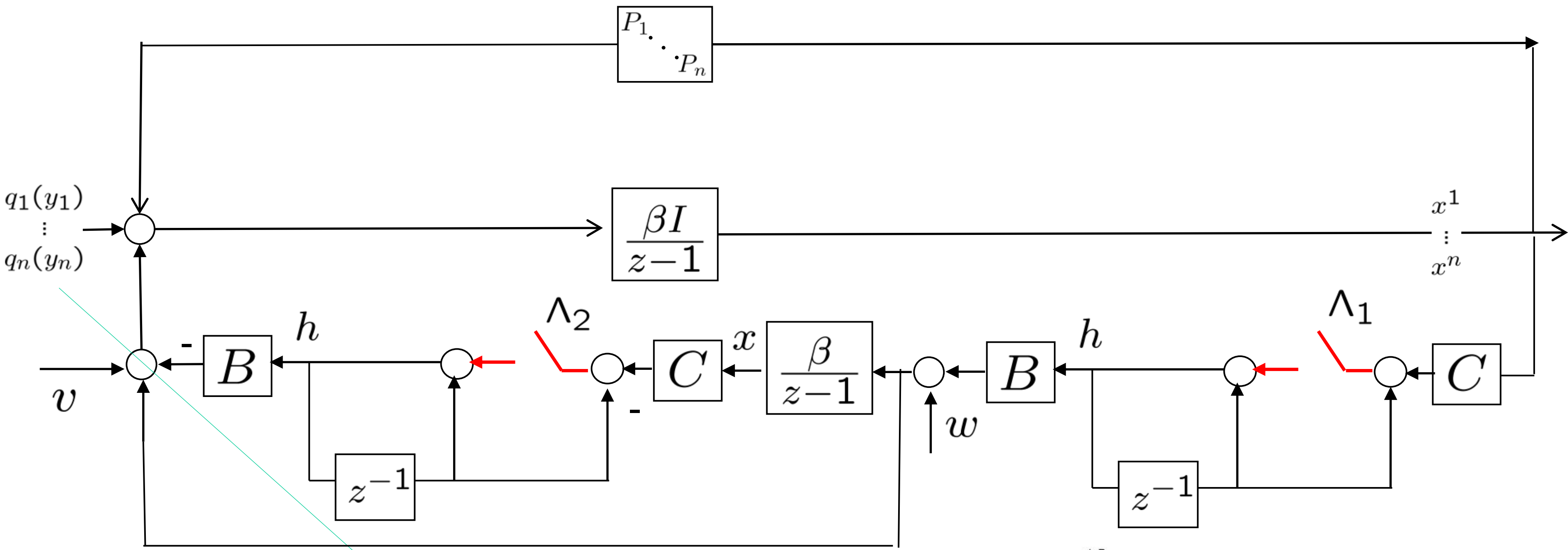
System is in a fragile state with high noise amplification

Unreliable Communication: a Mechanism for Emergent Behavior

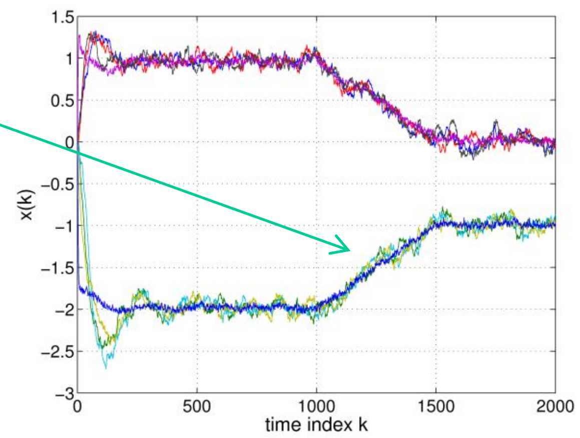
Constant speed
Averaging neighbors directions

Real-time Adaptive Optimization

[Wang, Elia., ACC12]



Changing measurements



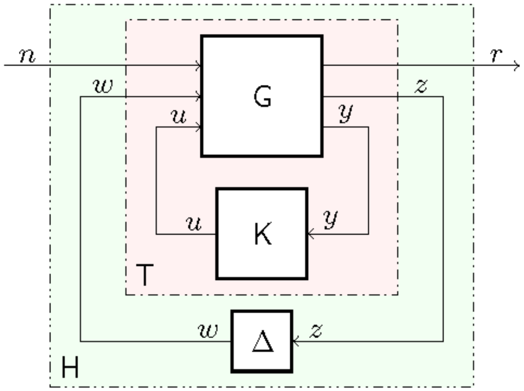
- Real-time adaptation to data
- Resilient to channel uncertainty
- Proof based on passivity theory

Robustness to noise and packet-drops

Joint work with Matt Rich

CONVEX MIMO DESIGN FOR MS PERFORMANCE OVER PACKET DROP NETWORKS

Analysis Extensions

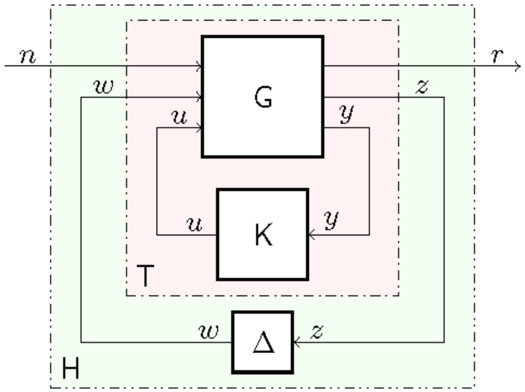


- Spatial correlation of stochastic uncertainty
- Still uncorrelated over time.
- MS Performance $\|H\|_{MS} = \nu$

For n(k) with $E\left(n(k)n^T(k)\right) = I$, let $R(k) = E\left(r(k)r^T(k)\right)$

H has MS performance ν if it is MS stable and $\lim_{k \to \infty} Tr\left(R(k)\right) = \nu^2$

MS Performance Analysis



$$T: \begin{bmatrix} x^+ \\ r \\ z \end{bmatrix} \left[\begin{array}{c|cc} \bar{A} & \bar{B}_n & \bar{B}_w \\ \hline \bar{C}_r & \bar{D}_{rn} & \bar{D}_{rw} \\ \bar{C}_z & \bar{D}_{zn} & \bar{D}_{zw} \end{array}\right] \begin{bmatrix} x \\ n \\ w \end{bmatrix}$$

- LMI characterization, specialize to spectral radius

$$\|H\|_{MS}^2 < \nu^2 \Leftrightarrow \exists\ (\mathbf{X},\mathbf{R},\mathbf{Z},\mathbf{W}) \in \mathbf{L} \text{ s.t. } \nu^2 = Tr(\mathbf{R})$$

$$\mathbf{X} \succ \bar{A}\mathbf{X}\bar{A}^T + \bar{B}_n\bar{B}_n^T + \bar{B}_w\mathbf{W}\bar{B}_w^T$$

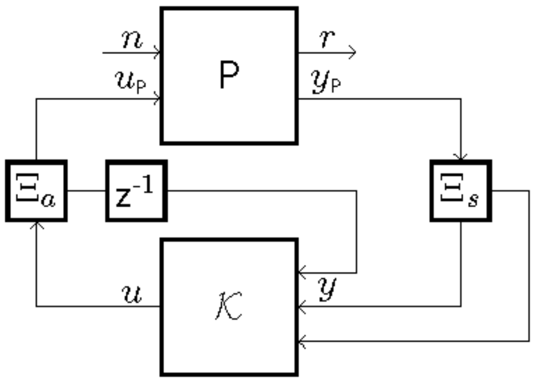
$$\mathbf{R} \succ \bar{C}_r\mathbf{X}\bar{C}_r^T + \bar{D}_{rn}\bar{D}_{rn}^T + \bar{D}_{rw}\mathbf{W}\bar{D}_{rw}^T$$

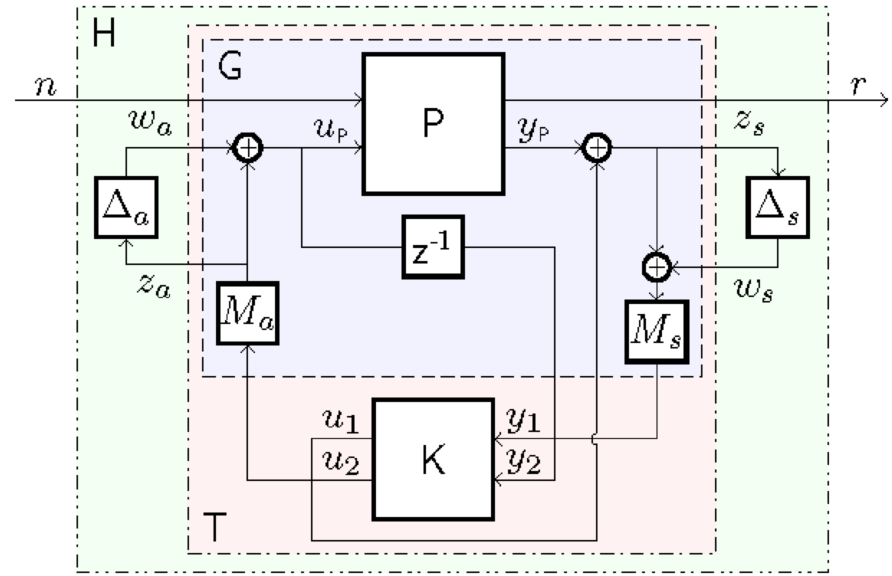
$$\mathbf{Z} \succ \bar{C}_z\mathbf{X}\bar{C}_z^T + \bar{D}_{zn}\bar{D}_{zn}^T + \bar{D}_{zw}\mathbf{W}\bar{D}_{zw}^T$$

$$\mathbf{W} \succ \Sigma \circ \mathbf{Z}$$

Σ Spatial correlation matrix of Δk

Optimal controller design problem

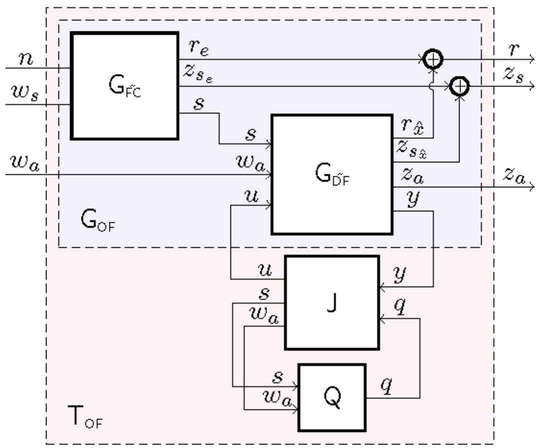




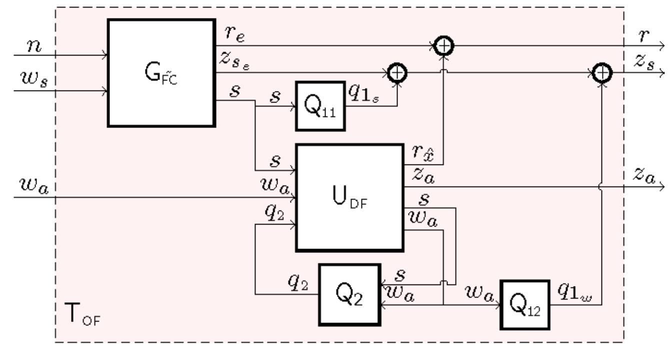
Deign an optimal controller
that minimizes the MS performance
from n -> r, in the presence of stochastic
packet drop uncertainty

$$
\mathsf{G}_{\mathsf{OF}}: \begin{bmatrix} x_{\mathsf{G}}^{+} \\ r \\ z_s \\ z_a \\ y_s \\ y_a \end{bmatrix} = \left[\begin{array}{c|ccc} A & B_n & \begin{bmatrix} 0 & B_w \end{bmatrix} & \begin{bmatrix} 0 & B_u \end{bmatrix} \\ \hline C_r & D_{rn} & \begin{bmatrix} 0 & D_{rw} \end{bmatrix} & \begin{bmatrix} 0 & D_r \end{bmatrix} \\ \begin{bmatrix} C_z \\ 0 \end{bmatrix} & \begin{bmatrix} D_{zn} \\ 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \\ \begin{bmatrix} C_y \\ 0 \end{bmatrix} & \begin{bmatrix} D_n \\ 0 \end{bmatrix} & \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}\right] \begin{bmatrix} x_{\mathsf{G}} \\ n \\ w_s \\ w_a \\ u_s \\ u_a \end{bmatrix}
$$

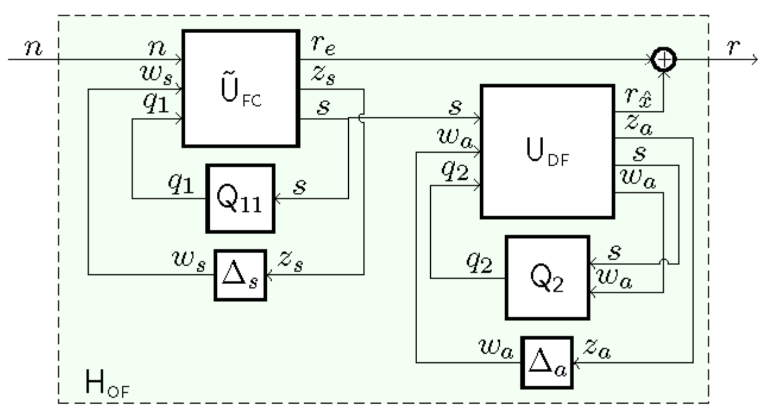
Idea of proof in pictures



General separation of closed loop maps



Simplification

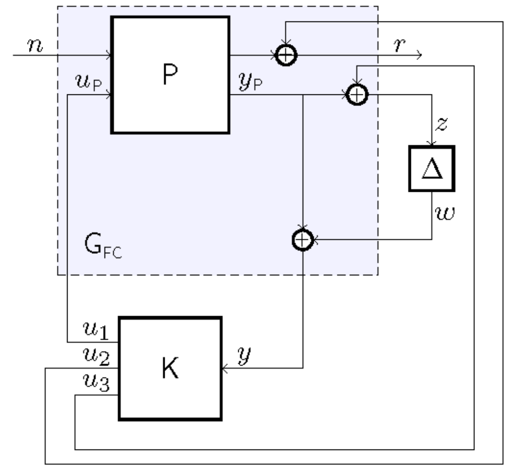


Separation of controller design

Separation Structure

Optimal Full Control problem

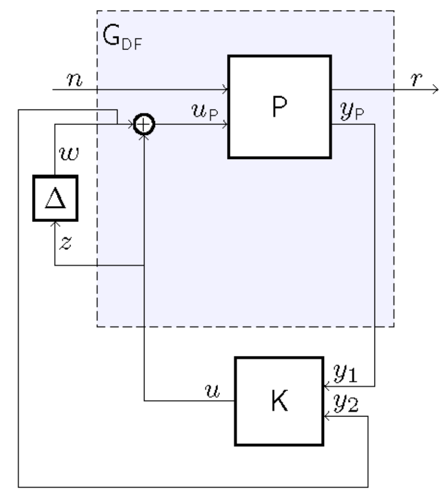
$$
\begin{aligned}
X_{FC} &= A X_{FC} A^T + B_n B_n^T - A X_{FC} C_y^T S_{FC}^{-1} C_y X_{FC} A^T \\
R_{FC} &= C_r X_{FC} C_r^T + D_{rn} D_{rn}^T - C_r X_{FC} C_y^T S_{FC}^{-1} C_y X_{FC} C_r^T \\
S_{FC} &= C_y X_{FC} C_y^T + D_n D_n^T + \Sigma_s \circ (C_z X_{FC} C_z^T + D_{zn} D_{zn}^T) \\
L^\star &= -A X_{FC} C_y^T S_{FC}^{-1} \\
L_0^\star &= -C_r X_{FC} C_y^T S_{FC}^{-1}
\end{aligned}
$$



(a) Networked Full Control

Optimal Disturbance Feed-forward problem

$$
\begin{aligned}
X_{DF} &= A^T X_{DF} A + C_r^T C_r - A^T X_{DF} B_u S_{DF}^{-1} B_u^T X_{DF} A \\
R_{DF} &= L^{\star T} X_{DF} L^\star + L_0^{\star T} L_0^\star - L^{\star T} X_{DF} B_u S_{DF}^{-1} B_u^T X_{DF} L^\star \\
S_{DF} &= B_u^T X_{DF} B_u + D_r^T D_r + \Sigma_a \circ (B_w^T X_{DF} B_w + D_{rw}^T D_{rw}) \\
F^\star &= -S_{DF}^{-1} B_u^T X_{DF} A \\
F_0^\star &= S_{DF}^{-1} B_u^T X_{DF} L^\star
\end{aligned}
$$



(b) Networked Disturbance Feedforward

Optimal MS Performance Design

$$(\nu^\star_{OF})^2 = \|\,\mathrm{F}_\ell(\mathrm{F}_\ell(\mathsf{G}_{\mathsf{OF}}, \mathsf{K}^\star_{\mathsf{OF}}), \Delta)\|^2_{MSP} = \mathrm{tr}(R_{FC}) + \mathrm{tr}(S_{FC} R_{DF})$$

where $\mathsf{K}^\star_{\mathsf{OF}}$ *has realization*

$$\mathsf{K}^\star_{\mathsf{OF}}: \begin{bmatrix} x^+_{\mathsf{K_{of}}} \\ u_s \\ u_a \end{bmatrix} = \left[\begin{array}{c|cc} A^\star_{\mathsf{K_{of}}} & B_u F_0^\star - L^\star & B_w \\ \hline -C_z & 0 & 0 \\ F^\star - F_0^\star C_y & F_0^\star & 0 \end{array}\right] \begin{bmatrix} x_{\mathsf{K_{of}}} \\ y_s \\ y_a \end{bmatrix}$$

with $A^\star_{\mathsf{K_{of}}} = A + B_u F^\star + L^\star C_y - B_u F_0^\star C_y$. *Moreover,* $\mathrm{tr}(R_{FC}) = \|\mathsf{H}^\star_{\mathsf{FC}}\|^2_{MSP}$ *where* $\mathsf{H}^\star_{\mathsf{FC}} = \mathrm{F}_\ell(\mathsf{T}^\star_{\mathsf{FC}}, \Delta_s)$ *with*

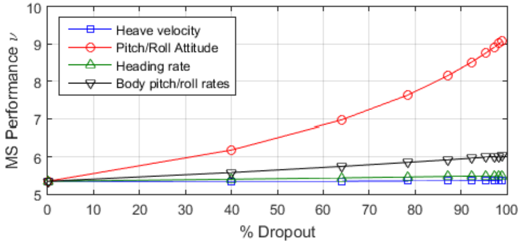
$$\mathsf{T}^\star_{\mathsf{FC}}: \begin{bmatrix} x^+_{\mathsf{fc}} \\ r_{\mathsf{fc}} \\ z_s \end{bmatrix} = \left[\begin{array}{c|cc} A + L^\star C_y & B_n + L^\star D_n & L^\star \\ \hline C_r + L_0^\star C_y & D_{rn} + L_0^\star D_n & L_0^\star \\ C_z & D_{zn} & 0 \end{array}\right] \begin{bmatrix} x_{\mathsf{fc}} \\ n \\ w_s \end{bmatrix}$$

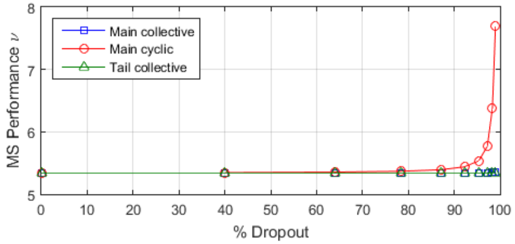
and $\mathrm{tr}(S_{FC} R_{DF}) = \|\mathsf{H}^\star_{\mathsf{DF}} S_{FC}^{\frac{1}{2}}\|^2_{MSP}$ *where* $\mathsf{H}^\star_{\mathsf{DF}} = \mathrm{F}_\ell(\mathsf{T}^\star_{\mathsf{DF}}, \Delta_s)$ *with*

$$\mathsf{T}^\star_{\mathsf{DF}}: \begin{bmatrix} x^+_{\mathsf{df}} \\ r_{\mathsf{df}} \\ z_a \end{bmatrix} = \left[\begin{array}{c|cc} A + B_u F^\star & B_u F_0^\star - L^\star & B_w \\ \hline C_r + D_r F^\star & D_r F_0^\star - L_0^\star & D_{rw} \\ F^\star & F_0^\star & 0 \end{array}\right] \begin{bmatrix} x_{\mathsf{df}} \\ s \\ w_a \end{bmatrix}$$

Remarks

- Controller switches based on the current/delayed channel states.
  - Leads to frequency domain tools.
  - Performance guarantees
- Controllers using Kalman filter with intermittent observations
  - Sample path dependent
  - Performance not know a priori
- Controllers using MJLS
  - More general but complex as depend on the collective state of channels





Conclusions

- Presented two streams of results at the interface of information and control theories
- Much progress has been made in the last 20 years
- There is still a robust set of open problems
- Need for engineering analysis and design tools
- New challenge: Incorporate learning theory

Acknowledgments

- To my students involved in the research I have presented

  Jailing Liu,

  Chong Li,

  Abhishek Rawat,

  Jing Wang,

  Xu Ma,

  Matt Rich.

- To NSF and AFOSR