Ergodicity and Asymptotic Stationarity of Controlled Stochastic Nonlinear Systems under Information Constraints

Nicolas Garcia - Joint work with Christoph Kawan and Serdar Yüksel

Queen's University, Dept. of Mathematics and Statistics

April 26, 2021

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- The techniques used build on the notion of invariance entropy for noiseless systems.
- I will be presenting some joint work with my supervisors Serdar Yüksel and Christoph Kawan, as well as some of their own past work.

Introduction

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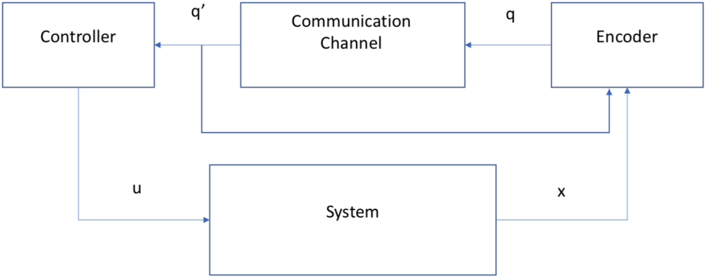

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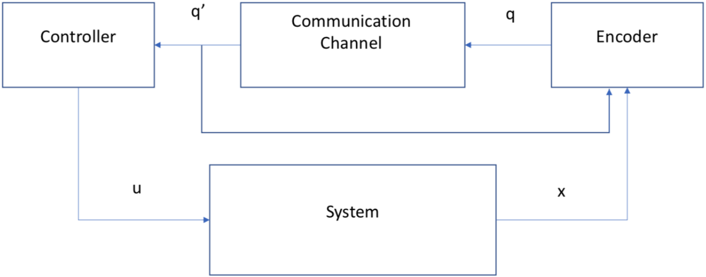

- The diagram depicts a channel with feedback. Of course, not all channels have this feature.

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- When a system is controlled over such a channel, a controller determines its action based on a *reliable estimate* of the system state.

Setup

System: We consider discrete-time non-linear systems of the form

$$x_{t+1} = f(x_t, w_t) + u_t \tag{1}$$

where (x_t), (u_t), and (w_t) are the state, control, and noise processes respectively.

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random variables $x_0, w_0, w_1, w_2, ...$ are all defined on a common probability space $(\Omega, \mathcal{F}, P)$.

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- Such a pair of sequences of functions is called a coding and control

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Then $x_1, x_2, x_3, , , ,$ and $u_0, u_1, u_2, ...$ are well defined random variables.Indeed:

$$u_0(\omega) = \gamma_0^c(\gamma_0^e(x_0(\omega))), \quad x_1(\omega) = f(x_0(\omega), w_0(\omega)) + u_0(\omega) \tag{3}$$

with similar definitions for larger time indices.

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$$x(\omega) \quad u(\omega) \quad w(\omega) \tag{4}$$

to denote the resulting state, control, and noise sequences given $\omega \in \Omega$.

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Before proceeding, some history.

- For linear systems such as

$$x_{t+1} = Ax_t + Bu_t \quad \text{and} \quad \dot{x}(t) = Ax(t) + Bu(t) \quad A \in \mathbb{R}^{n\times n} \tag{5}$$

it has been established (under many different assumptions and stability criteria) that the minimum data rate required for stabilization is the sum of logarithms of unstable eigenvalues of A.

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- Invariance entropy was introduced in [4] for the same problem, but in continuous time. When modified to the discrete time setting, this notion coincides with TFE.

1 2 3 4

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- In the latter two, stabilization entropy was used. This notion is a modification of invariance entropy for discrete-time stochastic systems and will be explained in detail later in the talk.

5 6 7

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- μ is known as the process measure corresponding to $(X_n)_{n\in\mathbb{N}}$.

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- It is not hard to check that $\mathcal{F}_{inv(\theta)}$ is itself a σ-algebra on Σ.

Stochastic Stability (Continued)

A few more definitions:

- Let $\theta : \Sigma \to \Sigma$ denote the left shift map on the sequence space (I.e. $\theta((x_t))_n = x_{n+1}$ for $(x_t) \in \Sigma$).
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- It is not hard to check that $\mathcal{F}_{inv(\theta)}$ is itself a σ-algebra on Σ.
- We are ready to define distinct notions of stochastic stability.

Stationary Processes

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(Stationarity) Let $(X_n)_{n\in\mathbb{N}}$ be a stochastic process taking values in $\mathbb{R}^N$ and let μ be its process measure on $\mathcal{B}(\Sigma)$.

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- It is immediate that stationarity implies AMS.
- Not hard to show that the AMS mean Q is stationary.

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- Ergodicity is an non-decomposability condition.
- It tells us that in a certain sense, the long term behavior of all sample paths is the same.

Example: Frequency of visits to a set.

Ergodicity and Asymptotic Ergodicity (Continued)

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$$A := \{(x_n) \in (\mathbb{R}^N)^{\mathbb{N}} : \lim_{T\to\infty} \frac{1}{T} \sum_{k=0}^{T-1} \mathbb{1}_B(x_k) = c\} \tag{10}$$

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We have all the background to state the main theorems discussed in this talk.

Results

The following are two theorems which provide lower bounds on channel capacity for ergodic and AMS stabilization.

Theorem 5

Consider the control system

$$x_{t+1} = f(x_t) + w_t + u_t \tag{11}$$

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Then if some technical assumptions are satisfied, we must have that

$$Q(B)\log_2\left(\inf_{x\in B}|\det Df(x)|\right) \leq C \tag{12}$$

for any $B \in \mathbb{R}^N$ *with finite and non-zero Lebesgue measure.*

Results (Continued)

Theorem 6

Consider the control system

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Under technical assumptions, we must have that

$$\int \int \log_2 |\det Df_w(x)| \, \mathrm{d}Q(x) \, \mathrm{d}\nu(w) \leq C. \tag{14}$$

where f_w *denotes the map* $x \mapsto f(x, w)$ *for a fixed noise symbol* w.

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Note: For both of the stated theorems, an identical result can be proven for scalar systems controlled over Discrete Memoryless Channels.

Intuition for the Bound

Consider the linear system

$$x_{t+1} = 5x_t + u_t \tag{15}$$

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- We know $x_0 \in [-1, 1]$ therefore we can quantize the state in bins

$$[-1, -0.6), [-0.6, -0.2), [-0.2, 0.2), [0.2, 0.6), [0.6, 1] \tag{16}$$

and apply the respective control decisions:

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Thus if controlled over a noiseless channel, we need a capacity of at least $\log_2(5)$ bits to accomplish this.For a non-linear system whose state trajectories are ergodic with measure Q, we might expect that we should average the logarithm of the Jacobian determinant of the system dynamics function w.r.t. the measure Q.

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- Suppose S_T is the smallest cardinality set of open-loop control sequences of length T with which we can render B invariant for the first T time steps.
- By the assumption that B can be rendered invariant using closed-loop control, we must have that

$$|S_T| \leq 2^{CT}. \tag{18}$$

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It follows from $|S_T| \leq 2^{CT}$ that

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- The quantity $\limsup_{T\to\infty}\frac{1}{T}\log_2|S_T|$ is known as the Invariance Entropy for the set B.

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- For this notion, it is only required that the rate at which a set B is visited be above some threshold.

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Notation: if we fix an initial state x_0, a noise sequence $w := (w_t)$, and control sequence $u := (u_t)$, the state trajectory (x_t) is uniquely defined. I.e.

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Digression: The second condition tells us the following. Let $\omega \in \tilde{\Omega}$. Then there exists $u \in S$ such that when we iterate the initial state $x_0(\omega)$ together with the sequences u and $w(\omega)$, the rate of state visits to B is no smaller than r on the time interval $0, 1, .., T-1$.

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The notion of stabilization entropy is crucial in the proofs of the two main theorems stated earlier.

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Then for any $B \in \mathcal{B}(\mathbb{R}^N)$ with finite and non-zero Lebesgue measure, we have that

$$Q(B) \log_2(\inf_{x \in B} |\det Df(x)|) \leq C.$$

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Lemma 10

For every $\epsilon > 0$ sufficiently small and any probability $\rho \in (0,1)$ we have that

$$h(B, Q(B) - \epsilon, \rho) \leq C. \tag{24}$$

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Note that at time T, the coding and control policy can generate no more than $|\mathcal{M}|^T$ distinct control sequences.As such, $|S_T| \leq |\mathcal{M}|^T$ and we obtain

$$h(B, Q(B) - \epsilon, \rho) \leq \limsup_{T\to\infty}(\frac{1}{T}\log_2 |S_T|) \tag{27}$$

$$\leq \limsup_{T\to\infty}(\frac{1}{T}\log_2 |\mathcal{M}|^T) = \log_2 |\mathcal{M}| = C \tag{28}$$

which completes the proof of the lemma.

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We will indeed find a suitable L which results in

$$Q(B) \log_2(\inf_{x\in B} |\det Df(x)|) \leq C. \tag{30}$$

for the set $B \subseteq \mathbb{R}^N$ that was previously fixed.

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- Decomposing it into 'fibers' and using Fubini-Tonelli, one can obtain

$$0 < \frac{\rho}{M} \leq |S_T| \max_{u \in S_T} \int m(A(u,w)) d\nu(w). \tag{31}$$

where

$$A(u,w) := \{x \in \mathbb{R}^N : \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{1}_B(\varphi(t,x,u,w)) \geq Q(B) - \epsilon\}.$$

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In words: $A(u, w)$ consists of initial conditions that, when paired with sequences u and w, visit the B at a rate no smaller than $Q(B) - \epsilon$ during the first T time steps.

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$$m(A(u,w))(\inf_{x\in B} |\det Df(x)|)^{(Q(B)-\epsilon)T} \leq m(\varphi_{T-1,u,w}(A(u,w))). \tag{35}$$

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- By doing this with as few partitions and bounding each one, we obtain

$$0 < \frac{\rho}{M} \leq |S_T| \max_{u \in S_T} \int m(A(u, w)) d\nu(w)$$

$$\leq |S_T| \cdot m(B)((Q(B) - \epsilon)T)^{-(Q(B)-\epsilon)T}.$$

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$$\frac{\rho}{M \cdot m(B)} \big(\inf_{x \in B} |\det Df(x)| \big)^{(Q(B)-\epsilon)T} \leq |S_T|.$$

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$$Q(B)\log_2(\inf_{x \in B} |\det Df(x)|) \leq \lim_{T \to \infty} \frac{1}{T}\log_2 |S_T| \leq C$$

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as desired. This completes the proof of the simplified theorem.

Generalization

Recall that we previously claimed that under asymptotic ergodicity, we obtain the bound

$$\int \int \log_2(|\det Df_w(x)|) dQ(x) dv(w) \leq C \qquad (36)$$

for the system $x_{t+1} = f(x_t, w_t) + u_t$.

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- As this holds for arbitrarily fine partitions, we can approximate the integral from below with the above bounds, from which the result follows.

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- If the desired channel capacity inequalities do not hold, one shows that ergodic/AMS stabilization is not possible, since if it were, one would be able to reconstruct x_0 with non-vanishing probability.
- This results in a contradiction with the strong converse theorem for DMCs.

Concluding Remarks

- As mentioned in the history section, bounds of the form

$$\int \int \log_2(|\det Df_w(x)|)dQ(x)dv(w) \leq C \tag{38}$$

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- Such bounds rely on inequalities involving differential entropy and mutual information.
- This approach is better suited for dealing with a larger class of communication channels, but cannot deal with systems for which state variables do not admit differential entropies.

Concluding Remarks

- Finally, we note that a recent refinement (using stabilization entropy tools) to the bound

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- This allows one to eliminate 'volume contracting' directions, and results in a tighter bound.
- This refinement is not possible with information theoretic techniques, where one cannot 'decouple' distinct coordinates of the system state-space due to non-zero mutual information between the distinct coordinates.
- For a state space decomposing into 'stable' and 'unstable' components, the bound becomes

$$\int \int \log_2(|\det D_{x^u} f_w(x^u, x^s)|) dQ(x^u, x^s) dv(w) \leq C \tag{40}$$

where the Jacobian determinant is a square matrix of size $< N$.

THE END