Feedback Capacity of Gaussian channels and Regret-based Control

Oron Sabag, Caltech

Online Seminar on Control and Information
May 10, 2021

Outline

1. Regret-optimal control

 - This part is based on joint work w. Gautam Goel, Sahin Lale and Babak Hassibi

2. The feedback capacity of Gaussian channels

 - This part is based on joint work w. Victoria Kostina and Babak Hassibi

The LQR setting

- A time-invariant linear dynamical system is given by

$$x_{t+1} = Ax_t + B_u u_t + B_w w_t,$$

  where $x_t \in \mathbb{R}^n$ is state, $u_t \in \mathbb{R}^m$ is the control and $w_t \in \mathbb{R}^p$ is the disturbance vector

- The pair (A, B_u) is stabilizable

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The operation:

- A policy $\mathcal{K}$ is a linear operator from $w = \{w_t\}$ to $u = \{u_t\}$
  - A causal mapping is a sequence of mappings

$$K_t : (w_{-\infty}, \ldots, w_t) \to u_t$$

  - A strictly causal policy is

$$K_t : (w_{-\infty}, \ldots, w_{t-1}) \to u_t$$

The linear quadratic cost

- The LQR cost of a linear controller $\mathcal{K}$ is

$$\begin{aligned}\mathrm{cost}(\mathcal{K}; w) &= \sum_{t=-\infty}^{\infty} (x_t^* Q x_t + u_t^* R u_t) \\ &\triangleq w^* T_{\mathcal{K}}^* T_{\mathcal{K}} w\end{aligned}$$

where $Q, R \succ 0$ are weight matrices.

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  where $Q, R \succ 0$ are weight matrices.
- For a linear controller (policy) $\mathcal{K}$, we can always write

$$\begin{bmatrix} x \\ u \end{bmatrix} = \underbrace{\begin{bmatrix} \mathcal{F}\mathcal{K} + \mathcal{G} \\ \mathcal{K} \end{bmatrix}}_{T_{\mathcal{K}}} w. \tag{1}$$

Strategies to design a controller

- One aims minimize the cost

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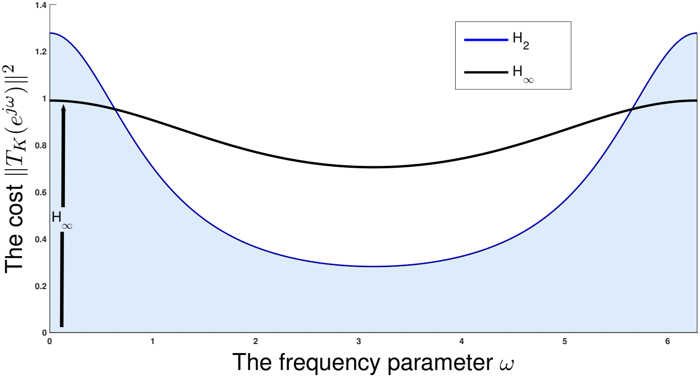
- The robust approach (H_∞ control):

$$\min_{\mathcal{K}} \max_{w \in \ell_2} \frac{\mathrm{cost}(\mathcal{K}; w)}{\|w\|_2}$$

Example



Scalar system with $A = 0.9$ and $B_u = B_w = Q = R = 1$.

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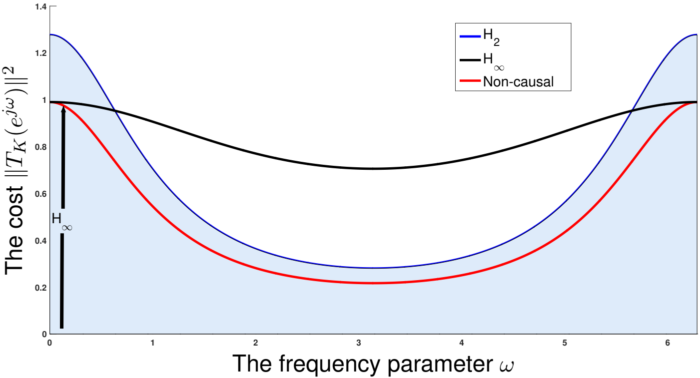

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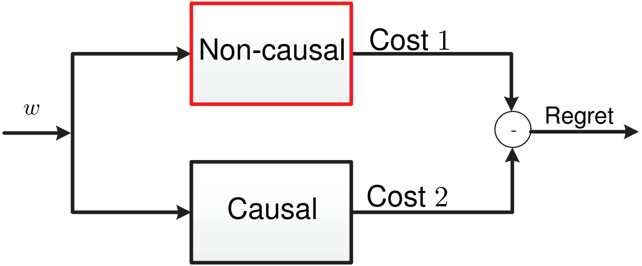
The Regret-Optimal Controller



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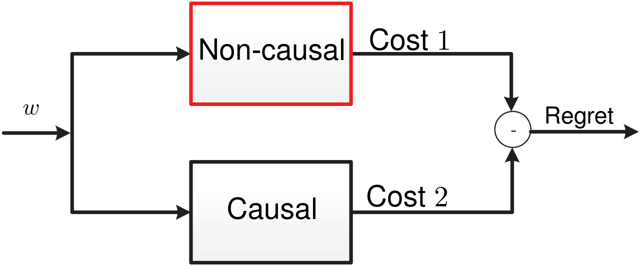

- Our regret approach:

$$\text{Regret}(\mathcal{K}; w) = \left(\text{cost}(\mathcal{K}; w) - \inf_{\mathcal{K}' \text{ is non-causal}} \text{cost}(\mathcal{K}'; w) \right)$$

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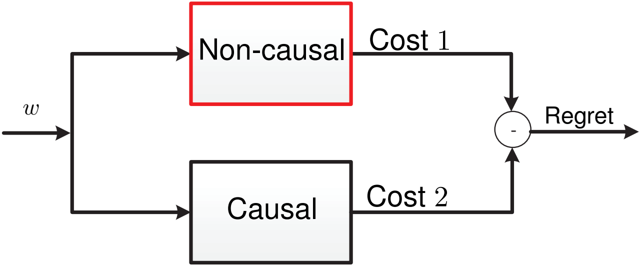

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- The design criterion is the worst-case regret:

$$\text{Regret}^* = \inf_{\mathcal{K} \text{ is causal}} \sup_{\|w\|_2 \leq 1} \text{Regret}(\mathcal{K};w).$$

Main results: the regret

Theorem (Sabag, Goel, Lale, Hassibi 21)

The optimal regret for the strictly-causal scenario is given by

$$\text{Regret}^* = \bar{\sigma}(Z\Pi), \tag{2}$$

where Z *and* Π *are the unique solutions for the Lyapunov equations*

$$\begin{aligned} Z &= A_K Z A_K^* + B_u(R + B_u^* P B_u)^{-1} B_u^* \\ \Pi &= A_K^* \Pi A_K + P B_w B_w^* P. \end{aligned} \tag{3}$$

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where P solves the LQR Riccati equation

$$\begin{aligned} P &= Q + A^*PA - A^*PB_u(R + B_u^*PB_u)^{-1}B_u^*PA \\ K_{lqr} &= (R + B_u^*PB_u)^{-1}B_u^*PA \\ A_K &= A - B_u K_{lqr} \end{aligned}$$

Main results: strictly-causal controller

Theorem (Sabag, Goel, Lale, Hassibi)

A strictly causal regret-optimal controller is given by

$$u_t = \hat{u}_t - K_{lqr} x_t, \quad (4)$$

where $\hat{u}_t$ is given by

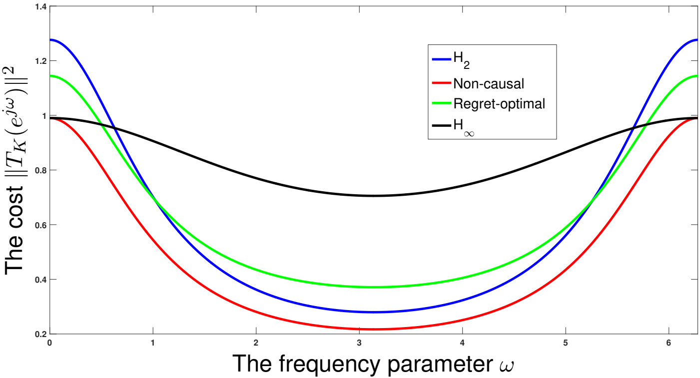
$$\begin{aligned} \xi_{t+1} &= F\xi_t + G w_t \\ \hat{u}_t &= -(R + B_u^* P B_u)^{-1} B_u^* \Pi \xi_t. \end{aligned} \quad (5)$$

and

$$\begin{aligned} G &= (I - A_K Z A_K^* \Pi)^{-1} A_K Z P B_w \\ F &= A_K - G B_w^* P, \end{aligned}$$

- Recall that $-K_{\text{lqr}} x_t$ is the standard LQR (H_2) controller

Example



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Comparison

	H_2 criterion (Frobenius)	H_∞ criterion (operator)
Noncausal	0.47	0.99
Regret-optimal	0.618	1.14
H_2 controller	0.598	1.28
H_∞ controller	0.84	0.99

Main ideas

- The regret can be reduced to a Nehari problem (1957)
  - Given an anticausal (upper triangular) operator $\mathcal{U}$,

$$\inf_{\mathcal{L} \text{ is causal}} \|\mathcal{L} - \mathcal{U}\|$$

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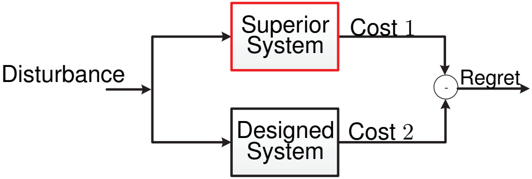
- The full-information control is just an example:



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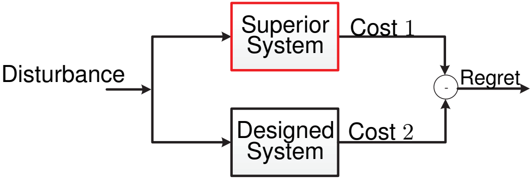
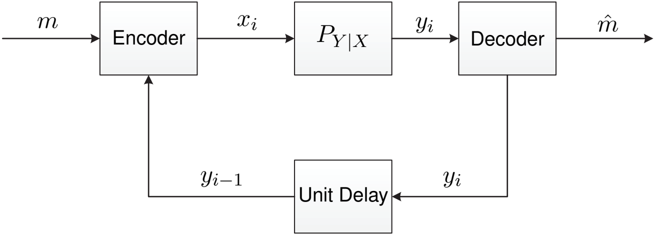


  - The filtering problem (Kalman setting) in AISTATS 2021

Part II: Feedback capacity of Gaussian channels

Channel with feedback



- A uniform message $m \in [1 : 2^{nR}]$
- At time i, encoding mapping is $e_i : [1 : 2^{nR}] \times \mathcal{Y}^{i-1}$
- Decoder mapping $\mathcal{Y}^n \to [1 : 2^{nR}]$

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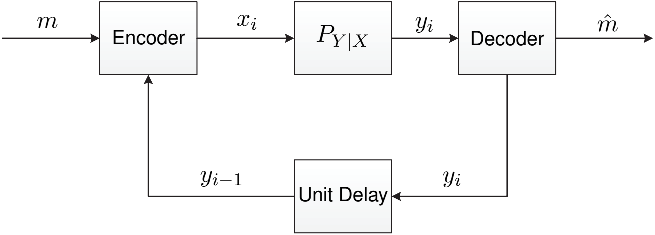

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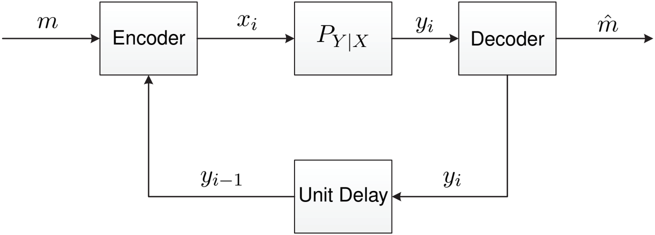

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- *Feedback does not increase the capacity* (Shannon 56)

- But, feedback has other benefits...

The AWGN channel

- The channel is given by

$$y_i = x_i + z_i,$$

where $\{z_i\}_{i\geq 1}$ is a white process with $z_i \sim N(0, Z)$

- An average power constraint $\frac{1}{n}\sum_{i=1}^{n} \mathrm{E}[x_i^2] \leq P$

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$$C_{fb}(P) = C(P) = \max I(X;Y) = 0.5 \log\left(1 + \frac{P}{Z}\right)$$

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1. Feedback does not increase the capacity

$$C_{fb}(P) = C(P) = \max I(X;Y) = 0.5 \log\left(1 + \frac{P}{Z}\right)$$

2. Feedback improves the probability of error

- In part, the linear Schalkwijk-Kailath (1966) coding

$$x_i \propto (z_0 - \hat{z}_0(y^{i-1}))$$

achieves doubly-exponential decay (as n grows)

The additive Gaussian noise channel

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- The current noise Z_i is correlated with Z^{i-1}

- An optimal input should exploit this correlation via Z^{i-1}

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  - The current noise Z_i is correlated with Z^{i-1}
  - An optimal input should exploit this correlation via Z^{i-1}
- The optimal input distribution is not i.i.d.
- *Feedback can increase the channel capacity*
  - But, not too much (Pinsker 69) (Ebert 70) (Cover-Pombra 89)

The first works

- Motivated by the SK scheme, Butman (67,69,76) studied $\{Z_i\}$ an auto-regressive (AR) noise

$$Z_i = \sum_{i=1}^{k} \alpha_i Z_{i-k} + U_i, \tag{6}$$

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- Achievable rates using linear coding schemes
- Upper bounds on the feedback capacity of AR noise

- Schemes and bounds also in Tiernan and Schalkwijk (74,76)

General capacity expression

Theorem (Cover, Pombra 89)

The feedback capacity of Gaussian channels is

$$C_{fb}(P) = \lim_{n\to\infty} \frac{1}{2n} \max_{B,\Sigma_V} \log \frac{\det \Sigma_{X+Z}^{(n)}}{\det \Sigma_Z^{(n)}}, \quad (7)$$

where the n*th maximization is over*

$$X^n = BZ^n + V^n$$

with B *being a strictly causal operator,* V^n *is a Gaussian process and*

$$\frac{1}{n}\mathrm{Tr}(\Sigma_X^{(n)}) \leq P.$$

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- For a fixed n, it is a convex program (Ordentlich, Boyd 98)
- Non-trivial to compute the limit

Past literature - I

- A. Dembo, ”On Gaussian feedback capacity,” 1989
- S. Ihara, ”Capacity of discrete time Gaussian channel with and without feedback-I,” 1988
- S. Ihara, ”Capacity of mismatched Gaussian channels with and without feedback,” 1990
- E. Ordentlich, ”A class of optimal coding schemes for moving average additive Gaussian noise channels with feedback,” 1994
- L. H. Ozarow, ”Random coding for additive Gaussian channels with feedback,” 1990.
- L. H. Ozarow, ”Upper bounds on the capacity of Gaussian channels with feedback,” 1990
- J. Wolfowitz, ”Signalling over a Gaussian channel with feedback and autoregressive noise,” 1975.
- L. Vandenberghe, S. Boyd, and S.-P. Wu, ”Determinant maximization with linear matrix inequality constraints,” 1998

The control perspective

- Yang-Kavcic-Tatikonda (2007) derive an MDP formulation
  - The formulation holds for any n
  - The MDP state is a covariance matrix
- For first-order ARMA,

$$Z_i + \beta Z_{i-1} = U_i + \alpha U_{i-1}, \text{ with } U_i \sim N(0,1) \tag{8}$$

they demonstrated the lower bound

$$C_{fb}(P) \geq -\log x_0,$$

and conjectured it to be the feedback capacity where x_0 is the positive root of $\frac{Px^2}{1-x^2} = \frac{(1+\sigma\alpha x)^2}{(1+\sigma\beta x)^2}$ with $\sigma = \text{sign}(\beta - \alpha)$

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- Kim (2006) confirms their conjecture for $\beta = 0$

Variational formula

- Kim (2009) - variational formula for stationary noise:

$$C_{\mathrm{FB}} = \sup_{S_V, B} \int_{-\pi}^{\pi} \frac{1}{2} \log \frac{S_V(e^{i\theta}) + |1 + B(e^{i\theta})|^2 S_Z(e^{i\theta})}{S_Z(e^{i\theta})} \frac{d\theta}{2\pi} \tag{6}$$

where $S_Z(e^{i\theta})$ is the power spectral density of the noise process $\{Z_i\}_{i=1}^{\infty}$ and the supremum is taken over all power spectral densities $S_V(e^{i\theta}) \geq 0$ and all strictly causal filters $B(e^{i\theta}) = \sum_{k=1}^{\infty} b_k e^{ik\theta}$ satisfying the power constraint

$$\int_{-\pi}^{\pi} (S_V(e^{i\theta}) + |B(e^{i\theta})|^2 S_Z(e^{i\theta})) \frac{d\theta}{2\pi} \leq P.$$

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- Still, not computable
- Resembles entropy in robust control (Mustafa, Glover 90), (Doyle, Glover 88)
- Computation of optimal S_V, B for ARMA noise of first order
  - This confirms the conjecture in (Yang et al. 07)

Past literature - II

- C. Li and N. Elia, ”Youla coding and computation of Gaussian feedback capacity,” 2018
- T. Liu and G. Han, ”Feedback capacity of stationary Gaussian channels further examined,” 2019
- C. D. Charalambous, C. K. Kourtellaris and S. Loyka ”Capacity achieving distributions and separation principle for feedback Gaussian channels with memory: the LQG theory of directed information,” 2018
- A. Gattami, ”Feedback capacity of Gaussian channels revisited,” 2019
- C. D. Charalambous, C. K. Kourtellaris and S. Loyka, ”New formulas of ergodic feedback capacity of AGN channels driven by stable and unstable autoregressive noise,” 2020
- S. Fang and Q. Zhu, ”A connection between feedback capacity and Kalman filter for colored Gaussian noises,” 2020

Our setting

- The channel is MIMO

$$\mathbf{y}_i = \Lambda \mathbf{x}_i + \mathbf{z}_i,$$

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$$\begin{aligned} \mathbf{s}_{i+1} &= F\mathbf{s}_i + G\mathbf{w}_i \\ \mathbf{z}_i &= H\mathbf{s}_i + \mathbf{v}_i, \end{aligned}$$

  where $(\mathbf{w}_i, \mathbf{v}_i) \sim N(0, \begin{pmatrix} W & L \\ L^T & V \end{pmatrix})$ is an i.i.d. sequence

- The initial state $s_1 \sim N(0, \Sigma_{1|0})$

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- The initial state $s_1 \sim N(0, \Sigma_{1|0})$
- When F (and $L = 0$) is stable, it is the *stationary case*

Reminder: Kalman filter

- Define

$$\hat{\mathbf{s}}_i = \mathrm{E}[\mathbf{s}_i | \mathbf{z}^{i-1}]$$
$$\Sigma_i = \mathbf{cov}(\mathbf{s}_i - \hat{\mathbf{s}}_i).$$

- The Kalman filter is given by

$$\hat{\mathbf{s}}_{i+1} = F\,\hat{\mathbf{s}}_i + K_{p,i}(\mathbf{z}_i - H\,\hat{\mathbf{s}}_i), \qquad (9)$$

with

$$K_{p,i} = (F\Sigma_i H^T + GL)\Psi_i^{-1}, \quad \Psi_i = H\Sigma_i H^T + V,$$

and the covariance update is

$$\Sigma_{i+1} = F\Sigma_i F^T + GWG^T - K_p\Psi_i K_p^T. \qquad (10)$$

- The *innovations process* is $\mathbf{e}_i = \mathbf{z}_i - H\,\hat{\mathbf{s}}_i$ with $\mathbf{e}_i \sim N(0, \Psi_i)$

The Riccati equation

- The recursion converges to the stabilizing solution of

$$\Sigma = F\Sigma F^T + W - K_p \Psi K_p^T,$$

  where $K_p = (F\Sigma H^T + GL)\Psi^{-1}$ and $\Psi = H\Sigma H^T + V$.
- In the stationary case, no further assumptions
- In the non-stationary case, we assume detectability and stabilizability

Main result

Theorem (Sabag, Kostina, Hassibi 21)

The feedback capacity of the MIMO Gaussian channel is

$$C^{fb}(P) = \max_{\Pi,\hat{\Sigma},\Gamma} \frac{1}{2}\log\det(\Psi_Y) - \frac{1}{2}\log\det(\Psi)$$

$$\Psi_Y = \Lambda\Pi\Lambda^T + H\hat{\Sigma}H^T + \Lambda\Gamma H^T + H\Gamma^T\Lambda^T + \Psi$$

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s.t. $\begin{pmatrix} \Pi & \Gamma \\ \Gamma^T & \hat{\Sigma} \end{pmatrix} \succeq 0, \quad \mathbf{Tr}(\Pi) \leq P,$

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$$\textit{s.t.} \quad \begin{pmatrix} \Pi & \Gamma \\ \Gamma^T & \hat{\Sigma} \end{pmatrix} \succeq 0, \quad \mathbf{Tr}(\Pi) \le P,$$

$$\begin{pmatrix} F\hat{\Sigma}F^T + K_p\Psi K_p^T - \hat{\Sigma} & F\Gamma^T\Lambda^T + F\hat{\Sigma}H^T + K_p\Psi \\ (\cdot)^T & \Psi_Y \end{pmatrix} \succeq 0$$

The channel:

$$\mathbf{y}_i = \Lambda\mathbf{x}_i + \mathbf{z}_i$$

The noise:

$$\mathbf{s}_{i+1} = F\mathbf{s}_i + G\mathbf{w}_i$$

$$\mathbf{z}_i = H\mathbf{s}_i + \mathbf{v}_i$$

The linear matrix inequalities (LMIs)

- The decision variable Π is the inputs covariance:
- The constraint $\mathbf{Tr}(\Pi) \leq P$ is the power constraint
- The first LMI

$$\begin{pmatrix} \Pi & \Gamma \\ \Gamma^T & \hat{\Sigma} \end{pmatrix} \succeq 0$$

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  corresponds to a Riccati inequality

$$\hat{\Sigma} \preceq F\hat{\Sigma}F^T + K_p\Psi K_p^T - (F\Gamma^T\Lambda^T + F\hat{\Sigma}H^T + K_p\Psi)\Psi_Y^{-1}(F\Gamma^T\Lambda^T + F\hat{\Sigma}H^T + K_p\Psi)^T$$

Main results: a scalar channel

Theorem

The feedback capacity of the scalar Gaussian channel is

$$C^{fb}(P) = \max_{\hat{\Sigma},\Gamma} \frac{1}{2}\log\left(1 + \frac{P + H\hat{\Sigma}H^T + 2\Gamma H^T}{\Psi}\right)$$

s.t. $\begin{pmatrix} P & \Gamma \\ \Gamma^T & \hat{\Sigma} \end{pmatrix} \succeq 0,$

$$\begin{pmatrix} F\hat{\Sigma}F^T + K_p\Psi K_p^T - \hat{\Sigma} & F\Gamma^T + F\hat{\Sigma}H^T + K_p\Psi \\ (F\Gamma^T + F\hat{\Sigma}H^T + K_p\Psi)^T & P + H\hat{\Sigma}H^T + 2\Gamma H^T + \Psi \end{pmatrix} \succeq 0,$$

where K_p *and* Ψ *are constants.*

- If $H = 0$, the capacity is $C(P) = \frac{1}{2}\log\left(1 + \frac{P}{V}\right)$.

Discussion

- This is the most general formulation with solution:
  1. General state-space
  2. Noise may be non-stationary
  3. MIMO channels
- The state-space structure is important
- The solution subsumes (Kim 06,09),

  and is *similar* to (Gattami 19) that studies a scalar channel with state-space that is stationary, controllable with fully-correlated disturbances

Can the capacity be simplified further?

The moving average noise

Consider $Z_i = U_i + \alpha U_{i-1}$ with $\alpha \in \mathbb{R}$ and $U_i \sim N(0,1)$

Theorem (Alternative expression for (Kim, 06))

The feedback capacity of first-order MA noise process is

$$C_{fb}(P) = \frac{1}{2}\log(1+\mathbf{SNR}), \tag{11}$$

where $\mathbf{SNR}$ *is the positive root of the polynomial* $\mathbf{SNR} = \left(\sqrt{P} + |\alpha|\sqrt{\frac{\mathbf{SNR}}{1+\mathbf{SNR}}}\right)^2$.

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- Proof: it is easy to show that the Schur complement of both LMIs equals zero. Substitute these equations into the objective.
- The fixed-point polynomial is different from (Kim 06)
  - However, their positive roots coincide

Main steps

Reminder: the capacity is given by

$$C_{fb}(P) = \lim_{n\to\infty} \frac{1}{2n} \max_{\{X_i = BZ^{i-1}+V_i\}_{i=1}^n} \log \frac{\det \Sigma_{X+Z}^{(n)}}{\det \Sigma_Z^{(n)}}$$

Road map:

1. *Sequentialize* the objective
2. *Sequentialize* the domain
3. Formulate a SCOP (sequential convex optimization problem)
4. A ”single-letter” upper bound
5. Show that the upper bound can be achieved

The directed information (DI)

1. The DI was defined in (Massey 90)

$$
\begin{aligned}
I(X^n \to Y^n) &= \sum_{i=1}^{n} I(X^i; Y_i | Y^{i-1}) \\
&= \sum h(Y_i|Y^{i-1}) - h(X_i + Z_i | Y^{i-1}, X^i, Z^{i-1}) \\
&= \sum h(Y_i|Y^{i-1}) - h(Z_i|Z^{i-1}) \\
&= h(Y^n) - h(Z^n)
\end{aligned}
$$

2. For Gaussian inputs, the *Cover and Pombra objective* is DI

$$
I(X^n \to Y^n) = \log \frac{\det K_{X+Z}^{(n)}}{\det K_{Z}^{(n)}} \tag{12}
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3. Aligns with feedback capacity theorems (Tatikonda, Mitter 00,09) (Permuter, Weissman, Goldsmith 08)

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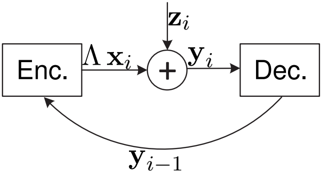
The structure: cascaded filtering problem



The encoder constructs
$\hat{\mathrm{s}}_i \triangleq \mathrm{E}[\mathbf{s}_i|\,\mathbf{z}^{i-1}]$ from

$$\mathbf{s}_{i+1} = F\mathbf{s}_i + G\mathbf{w}_i$$
$$\mathbf{z}_i = H\mathbf{s}_i + \mathbf{v}_i,$$

The innovation $\Psi_i = \text{cov}(\mathbf{z}_i - H\,\hat{\mathrm{s}}_i)$

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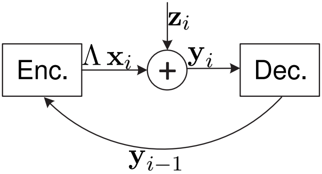

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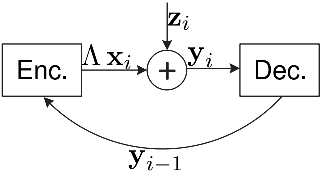

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- The objective reads

$$h(Y_i|Y^{i-1}) - h(Z_i|Z_{i-1}) = 0.5\log\det(\Psi_{Y,i}) - 0.5\log\det(\Psi_i)$$

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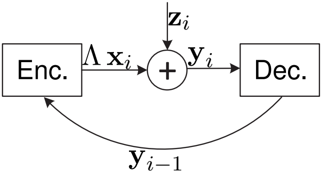

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The optimal policy

Lemma

For each n, it is sufficient to optimize with inputs of the form

$$\mathbf{x}_i = \Gamma_i \hat{\Sigma}_i^{\dagger} (\hat{\mathbf{s}}_i - \hat{\hat{\mathbf{s}}}_i) + \mathbf{m}_i, \quad i = 1, \ldots, n$$

where:

- Similar policy structures in (Yang et al. 07), (Kim 09), (Gattami 19), (Charalmbous et al. 18, 20)

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$$\Gamma_i (I - \hat{\Sigma}_i^{\dagger} \hat{\Sigma}_i) = 0$$

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- Γ_i *is a matrix that satisfies*

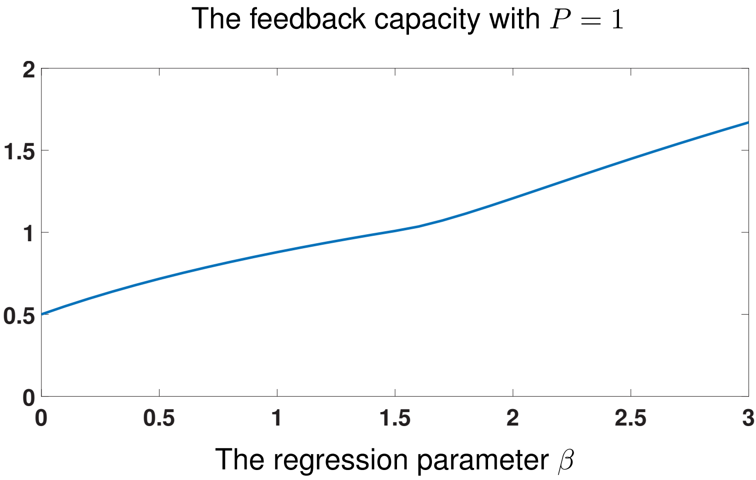
$$\Gamma_i(I - \hat{\Sigma}_i^{\dagger}\hat{\Sigma}_i) = 0$$

- *the input satisfies* $\sum_{i=1}^{n} \mathbf{Tr}(\Gamma_i \hat{\Sigma}_i^{\dagger} \Gamma_i^T + M_i) \leq nP$

- Similar policy structures in (Yang et al. 07), (Kim 09), (Gattami 19), (Charalmbous et al. 18, 20)

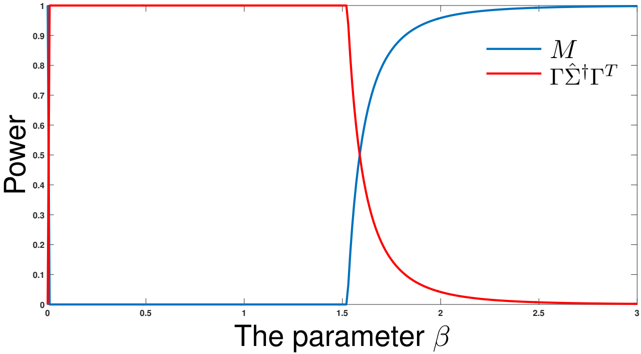
The AR noise

- Consider the AR noise $Z_i + \beta Z_{i-1} = U_i$ with $U_i \sim N(0,1)$



The AR noise - contd.

- The optimal inputs are $\mathbf{x}_i = \Gamma\hat{\Sigma}^{\dagger}(\hat{\mathbf{s}}_i - \hat{\hat{\mathbf{s}}}_i) + \mathbf{m}_i$
  - The power of each component



- The range $\beta \in [0, 1.5]$ shows our disagreement with (Gattami 19)
- For large β, i.i.d. inputs become optimal

Main steps

Reminder: the capacity is given by

$$C_{fb}(P) = \lim_{n\to\infty} \frac{1}{2n} \max_{\{X_i = BZ^{i-1}+V_i\}_{i=1}^n} \log \frac{\det \Sigma_{X+Z}^{(n)}}{\det \Sigma_Z^{(n)}}$$

Road map:

- ✓ *Sequentialize* the objective
- ✓ *Sequentialize* the domain

3. Formulate a SCOP
4. A "single-letter" upper bound
5. Show that the upper bound can be achieved

The controlled state-space

Lemma

For a fixed policy $\{(\Gamma_i, M_i)\}_{i=1}^n$,

$$\hat{\mathbf{s}}_{i+1} = F\,\hat{\mathbf{s}}_i + K_{p,i}\,\mathbf{e}_i,$$
$$\mathbf{y}_i = (\Lambda\Gamma_i\hat{\Sigma}_i^\dagger + H)\,\hat{\mathbf{s}}_i - \Lambda\Gamma_i\hat{\Sigma}_i^\dagger\,\hat{\hat{\mathbf{s}}}_i + \Lambda\mathbf{m}_i + \mathbf{e}_i,$$

Consequently, the error covariance $\hat{\Sigma}_i = \mathbf{cov}(\hat{\mathbf{s}}_i - \hat{\hat{\mathbf{s}}}_i)$ *satisfies*

$$\hat{\Sigma}_{i+1} = F\hat{\Sigma}_iF^T + K_{p,i}\Psi_iK_{p,i}^T - K_{Y,i}\Psi_{Y,i}K_{Y,i}^T$$

with $\hat{\Sigma}_1 = 0$, *and*

$$\Psi_{Y,i} = (\Lambda\Gamma_i\hat{\Sigma}_i^\dagger + H)\hat{\Sigma}_i(\Lambda\Gamma_i\hat{\Sigma}_i^\dagger + H)^T + \Lambda M_i\Lambda^T + \Psi_i$$
$$K_{Y,i} = (F\hat{\Sigma}_i(\Lambda\Gamma_i\hat{\Sigma}_i^\dagger + H)^T + K_{p,i}\Psi_i)\Psi_{Y,i}^{-1}$$

- Similar state-space in (Kim 09), (Charalmbous et al. 20)

SCOP formulation

Lemma (Sequential convex-optimization problem)

The n-letter capacity can be bounded as

$$C_n(P) \leq \max_{\{\Gamma_i, \Pi_i, \hat{\Sigma}_{i+1}\}_{i=1}^n} \frac{1}{2n} \sum_{i=1}^{n} \log\det(\Psi_{Y,i}) - \log\det(\Psi_i)$$

$$s.t. \quad \begin{pmatrix} \Pi_t & \Gamma_t \\ \Gamma_t^T & \hat{\Sigma}_t \end{pmatrix} \succeq 0, \quad \frac{1}{n} \sum_{i=1}^{n} \mathbf{Tr}(\Pi_i) \leq P,$$

$$\begin{pmatrix} F\hat{\Sigma}_t F^T + K_{p,t}\Psi_t K_{p,t}^T - \hat{\Sigma}_{t+1} & K_{Y,t}\Psi_{Y,t} \\ \Psi_{Y,t}K_{Y,t}^T & \Psi_{Y,t} \end{pmatrix} \succeq 0,$$

where the LMIs hold for $t = 1, \ldots, n$ and $\hat{\Sigma}_1 = 0$.

Proof outline

- The argument of the objective is

$$\Psi_{Y,i} = (\Lambda\Gamma_i\hat{\Sigma}_i^\dagger + H)\hat{\Sigma}_i(\Lambda\Gamma_i\hat{\Sigma}_i^\dagger + H)^T + \Lambda M_i\Lambda^T + \Psi_i$$

  - Define an auxiliary decision variable $\Pi_i \triangleq M_i + \Gamma_i\hat{\Sigma}_i^\dagger\Gamma^T$
  - Reduce the variable M_i
- The Schur complement transformation (e.g. Boyd 94)

$$\begin{matrix} \Pi_i \succeq \Gamma_i\hat{\Sigma}_i^\dagger\Gamma_i^T \\ \Gamma_i(I - \hat{\Sigma}_i^\dagger\hat{\Sigma}_i) = 0 \end{matrix} \iff \begin{pmatrix} \Pi_i & \Gamma_i \\ \Gamma_i^T & \hat{\Sigma}_i \end{pmatrix} \succeq 0.$$

- Relax Riccati recursion to a matrix inequality + Schur complement transformation

Single-letter Upper Bound

Lemma (The upper bound)

The feedback capacity is bounded by the convex optimization problem

$$C_{fb}(P) \leq \max_{\Pi,\hat{\Sigma},\Gamma} \frac{1}{2}\log\det(\Psi_Y) - \frac{1}{2}\log\det(\Psi)$$

$$\textit{s.t.} \quad \begin{pmatrix} \Pi & \Gamma \\ \Gamma^T & \hat{\Sigma} \end{pmatrix} \succeq 0, \quad \mathbf{Tr}(\Pi) \leq P,$$

$$\Psi_Y = \Lambda\Pi\Lambda^T + H\hat{\Sigma}H^T + \Lambda\Gamma H^T + H\Gamma^T\Lambda^T + \Psi$$

$$K_Y = (F\Gamma^T\Lambda^T + F\hat{\Sigma}H^T + K_p\Psi)\Psi_Y^{-1}$$

$$\begin{pmatrix} F\hat{\Sigma}F^T + K_p\Psi K_p^T - \hat{\Sigma} & K_Y\Psi_Y \\ \Psi_Y K_Y^T & \Psi_Y \end{pmatrix} \succeq 0.$$

Proof outline

- Define the uniform convex combinations

$$\bar{\Pi}_n = \frac{1}{n}\sum_{i=1}^{n}\Pi_i, \quad \bar{\Gamma}_n = \frac{1}{n}\sum_{i=1}^{n}\Gamma_i, \quad \bar{\hat{\Sigma}}_n = \frac{1}{n}\sum_{i=1}^{n}\hat{\Sigma}_i$$

- By the concavity of $\log\det(\cdot)$,

$$\frac{1}{n}\sum_{i=1}^{n}\log\det(\Psi_{Y,i}) \leq \log\det\left(\frac{1}{n}\sum_{i=1}^{n}\Psi_{Y,i}\right)$$

- Some of the constraints are satisfied for each n
- The Riccati LMI, however, is satisfied in the asymptotics only

Lower bound

Lemma (Lower bound)

The feedback capacity is lower bounded by the optimization problem

$$C_{fb}(P) \geq \max_{\Gamma,\Pi,\hat{\Sigma}} \log\det(\Psi_Y) - \log\det(\Psi)$$

$$\textit{s.t.} \quad \begin{pmatrix} \Pi & \Gamma \\ \Gamma^T & \hat{\Sigma} \end{pmatrix} \succeq 0, \quad \mathbf{Tr}(\Pi) \leq P$$

$$K_Y = (F\hat{\Sigma}H^T + F\Gamma^T\Lambda^T + K_p\Psi)\Psi_Y^{-1}$$

$$\Psi_Y = \Lambda\Pi\Lambda^T + \Lambda\Gamma H^T + H\Gamma^T\Lambda^T + \Psi$$

$$\hat{\Sigma} = F\hat{\Sigma}F^T + K_p\Psi K_p^T - K_Y\Psi_Y K_Y^T$$

$$\exists K : \rho(F - K(\Lambda\Gamma\hat{\Sigma}^\dagger + H)) < 1.$$

- Convergence of Riccati recursion (Nicolao, Gevers 92)

Conclusions

- A closed-form capacity expression as a finite-dimensional convex optimization problem
- The derivation relies on the noise state-space
- Sequential structures also exploited in (Tanaka, Kim, Parillo, Mitter 16) and its extension in (Sabag, Tian, Kostina, Hassibi 20)
- Ongoing work:
  - Optimal (and simple) coding scheme

Thank you for your attention!