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Quantized Control of Switched Linear Systems

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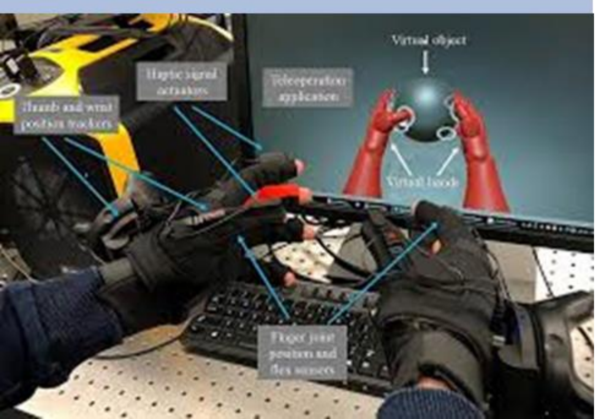
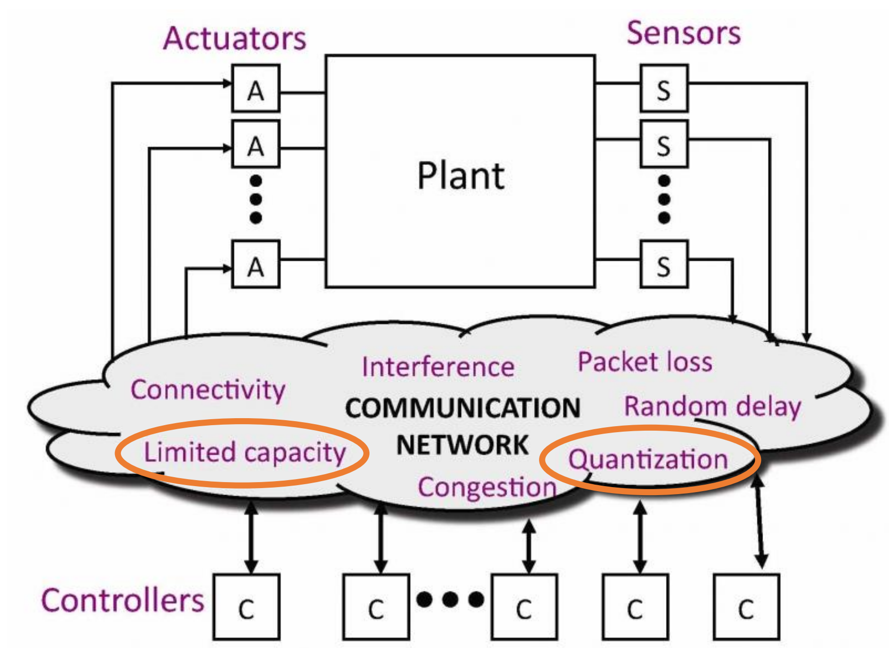
Joint work with **Guillaume Berger**
(Thanks also to **D. Liberzon and G. Yang** for many discussions during this research)

Virtual seminar on the interaction of control and information, 2021



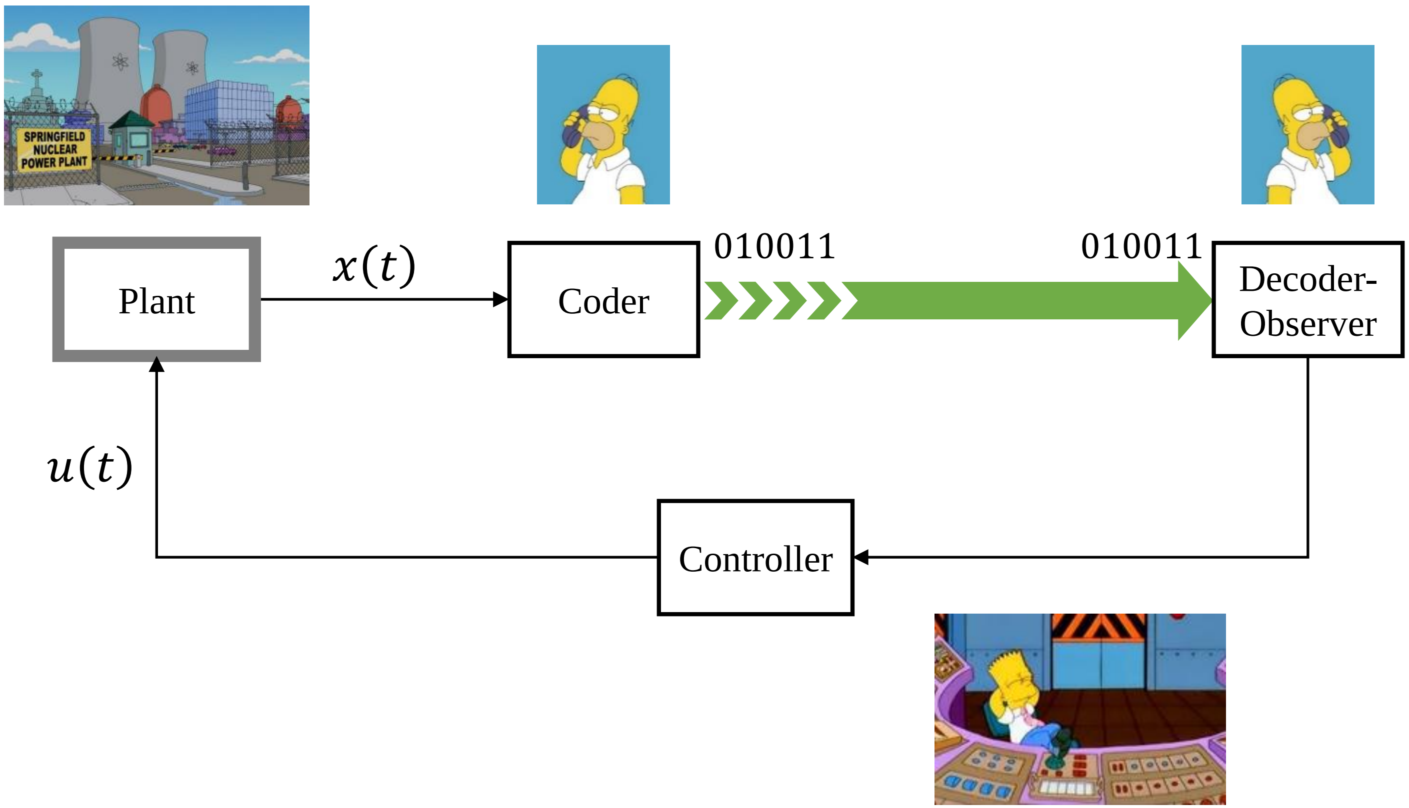
Networked control systems

Systems in which the different agents are spatially distributed and communicate through a digital communication network



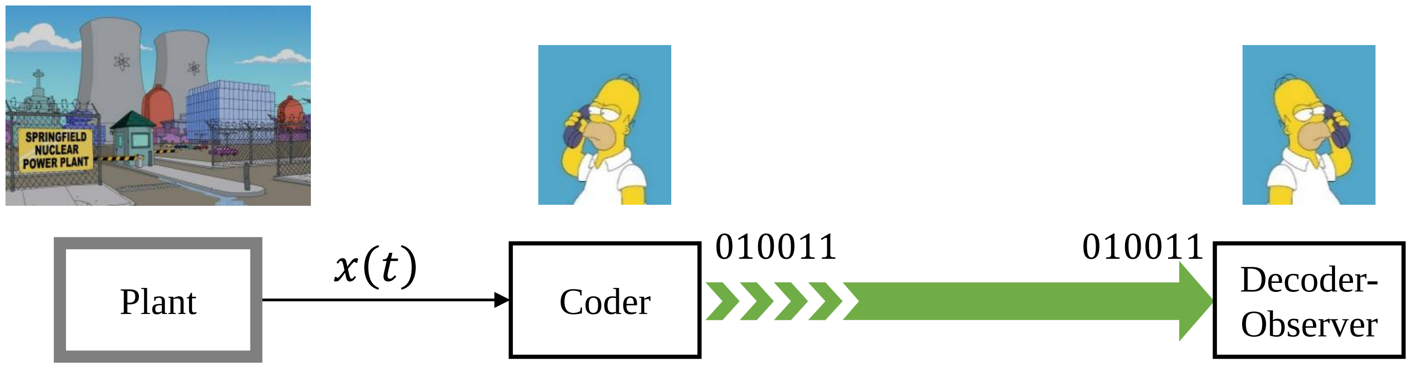
http://automatica.dei.unipd.it/people/schenato/research.html

Problem setting



- Minimal **data rate** for stabilization?
- Practical implementation of the **coder-decoder**?

Problem setting



Assumptions on plant and coder-decoder:

Time-invariant system
Time-varying system
Switched system
…

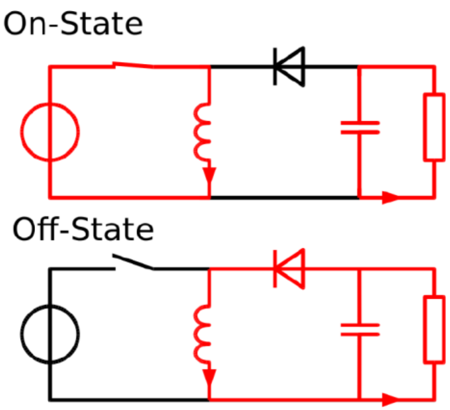
Mode-dependent coder-decoder
Mode-oblivious coder-decoder

Some pointers: [Brockett+2000], [Nair+2007], [Hespanha+2007], [Colonius+2008], [Matveev+2016], …, [Pogromsky+2011], [Kawan2017], [Berger+2020], …, [Liberzon2014], [Yang+2018], …

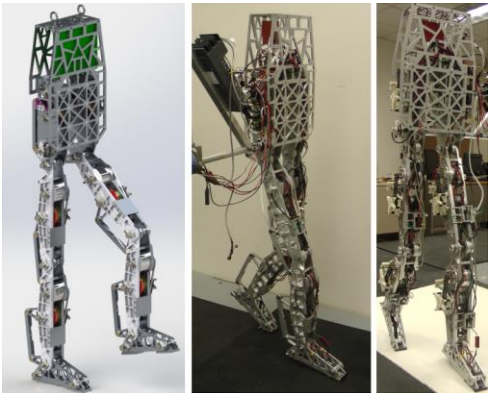
Switched systems

Systems of the form:

$$\dot{x}(t) = f_{\sigma(t)}\big(x(t), u(t)\big)$$



$\Sigma \coloneqq \{1, \dots, N\}$ *set of modes*
$\sigma : \mathbb{R}_+ \to \Sigma$ *switching signal*
$f_i : \mathbb{R}^d \times \mathbb{R}^c \to \mathbb{R}^d$ *continuous dynamics*



Switched *linear* systems:

$$\dot{x}(t) = A_{\sigma(t)} x(t) + B_{\sigma(t)} u(t)$$

$A_i \in \mathbb{R}^{d\times d}$, $B_i \in \mathbb{R}^{d\times c}$ *state/input transition matrices*

Outline of the presentation

1. Preliminaries and related works

2. Main results

    A. Stabilization with mode-dependent coder-controller

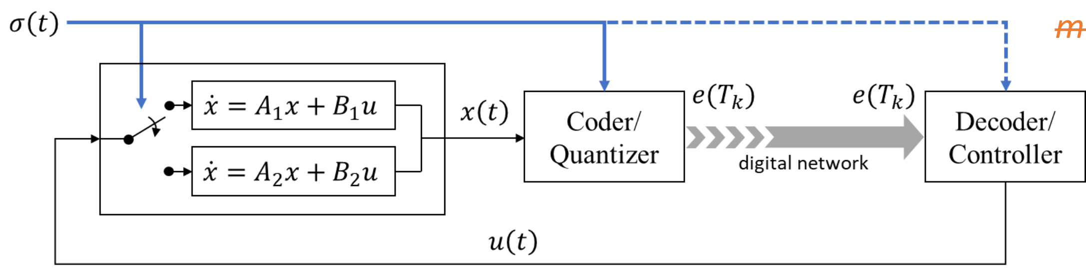
    B. Stabilization with mode-oblivious coder-controller

3. Some proofs of the main results and numerical examples

4. Conclusions

Coder-controller and data rate

mode-dependent
~~*mode-oblivious*~~



$T_k = k\tau_s \quad (k = 0,1,2,\ldots)$ — **transmission** times: periodic

$\mathcal{E}_k$ — **coding alphabet** at time T_k: finite size

$\gamma_k : \mathcal{X}^{k+1} \times \Sigma^{[0,T_k)} \to \mathcal{E}_k$ — **coder function** at time T_k

$e(T_k) = \gamma_k\big(x(T_0), \ldots, x(T_k), \sigma|_{[0,T_k)}\big)$ — **symbol** transmitted at time T_k

$\zeta_t : \mathcal{E}_0 \times \cdots \times \mathcal{E}_k \left[\times\, \Sigma^{[0,t)}\right] \to \mathbb{R}^c$ — **controller function** at time $t \in [T_k, T_{k+1})$

$u(t) = \zeta_t\big(e(T_0), \ldots, e(T_k) \left[, \sigma|_{[0,t)}\right]\big)$ — **control input** at time $t \in [T_k, T_{k+1})$

Data rate of the coder-controller:

$$R(\gamma,\zeta) = \sup_{k\in\mathbb{N}} \frac{\lceil \log_2 |\mathcal{E}_k| \rceil}{\tau_s}$$

$$\frac{\textit{binary size of transmitted symbol}}{\textit{interval between transmissions}}$$

Related works

Mode-dependent coder-controller

- Stabilizing **Markov Jump Linear Systems**:
  - Expression for the minimal **expected** data rate [Nair+2003]
  - Computable **lower bounds** and **upper bounds** on the data rate, and implementation [Zhang+2009, Ling+2010, Xiao+2010]
- **Observing discrete-time** switched linear systems: [Berger+2020]
  - Computable expression for the minimal data rate for observation
  - Practical implementation of the coder-observer
- **This work:** Stabilizing continuous-time switched linear systems

Related works

Mode-oblivious coder-controller

- Stabilizing continuous-time switched linear systems [Liberzon2014], with output [Wakaiki+2014], disturbance [Yang+2018]:
    - Sufficient relationship between the communication parameters (data rate, …) and the switching parameters (average dwell time, …) to ensure stabilization

- **This work:**
    - *Also:* Sufficient relationship between the communication parameters and the switching parameters to ensure stabilization
    - *But:* Focus on arbitrarily small switching parameters (namely ADT)
    - Show that non-zero ADT is necessary and sufficient for stabilization

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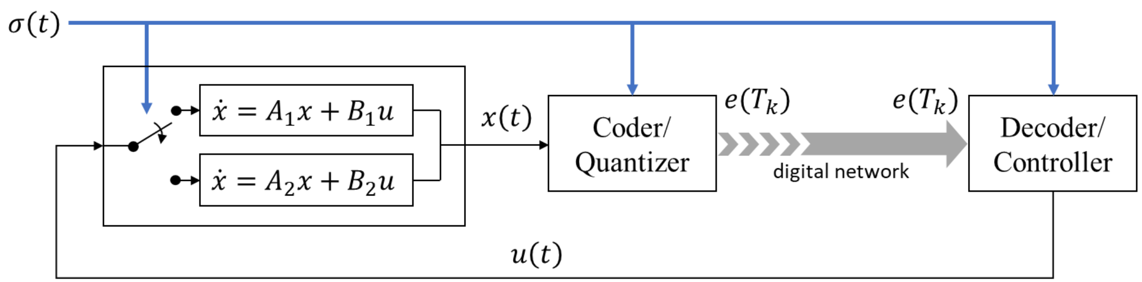
Mode-dependent coder-controller



Stabilizing $\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t)$ *with <u>mode-dependent</u> coder-controller*

Assumption: We assume that the system is stabilizable in the absence of data-rate constraints:

*(clearly: it is a **necessary** condition for being stabilizable with data-rate constraints)*

There is a **feedback law** $\varphi : \mathbb{R}^d \times \Sigma \to \mathbb{R}^c$ such that the closed-loop system with $u(t) \coloneqq \varphi\big(x(t), \sigma(t)\big)$ satisfies

$$\|x(t)\| \leq D\|x(0)\|e^{-\mu t}$$

for some $D \geq 0$, $\mu > 0$, and for all switching signals

Mode-dependent coder-controller

We use tools from:

- Control theory: **Lyapunov exponent** of switched linear systems: $\lambda(A_\Sigma)$
- Multilinear algebra: **exterior power** of linear maps: $A_i^{\odot}$

Computable closed-form expression for the optimal data rate:

Main result #1: The **minimal data rate** R_* for stabilization of switched linear systems with a mode-dependent coder-controller is

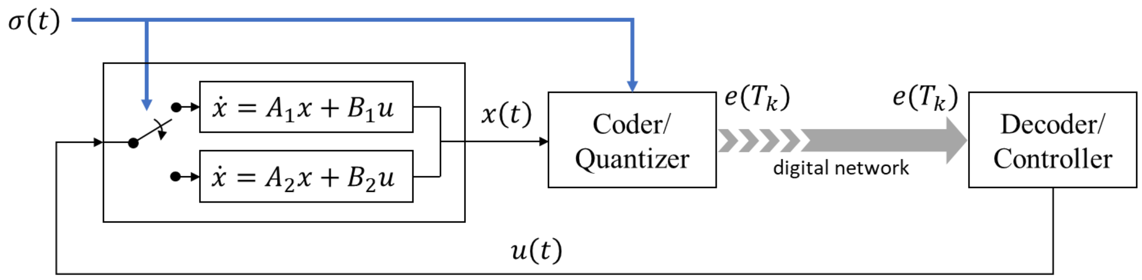
$$R_* = \log_2(e)\,\lambda\big(A_\Sigma^{\odot}\big)$$

… *generalizes the well-known formula for **LTI** systems:* $R_* = \log_2(e)\sum_{\Re(\lambda_i(A))>0}\lambda_i(A)$

Practical attainability of the optimal data rate:

Main result #2: There is a **practical** (i.e., implementable) mode-dependent coder-controller with data rate **arbitrarily close** to the minimal data rate, that stabilizes the system

Mode-oblivious coder-controller



Stabilizing $\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t)$ *with* *mode-oblivious* *coder-controller*

Assumption #1: There is an **average dwell time** (ADT) $\tau_a > 0$:

$$N_\sigma(s,t) \coloneqq \#\text{ switches of }\sigma\text{ in }[s,t) \leq C + \frac{t-s}{\tau_a}$$

Assumption #2: We assume that the system is stabilizable in the absence of data-rate constraints:

There is a **feedback law** $\varphi : \mathbb{R}^d \times \Sigma \to \mathbb{R}^c$ such that the closed-loop system with $u(t) \coloneqq \varphi\big(x(t), \sigma(t)\big)$ satisfies

$$\|x(t)\| \leq D\|x(0)\|e^{\mu_1 N_\sigma(0,t) - \mu_2 t}, \qquad \mu_1/\tau_a < \mu_2$$

Mode-oblivious coder-controller

Necessity of Assumption #1 (ADT>0):

Main result #1: Continuous-time switched linear systems with **zero ADT** are in general **not stabilizable** with a mode-oblivious coder-decoder with **finite data rate**

Existence of practical coder-controller:

Main result #2: Under the standing assumptions, there is a **practical** mode-oblivious coder-decoder with **finite data rate** that exponentially stabilizes the system: i.e., there is $\lambda > 0$ and a $\mathcal{K}$-function $g(\cdot)$, such that the closed-loop system satisfies

$$\|x(t)\| \leq g(\|x(0)\|)e^{-\lambda t}$$

Comparison of the settings

	Mode-dependent	Mode-oblivious
Analysis	*Stronger results* Closed-form expression for the optimal data rate Practical attainability of the optimal data rate	*Weaker results* Only upper bounds on the optimal data rate (depending on ADT, assuming ADT>0)
Application	*More requirements* Well suited for event-triggered quantized control Fundamental lower bounds on data rate for other coders-controllers	*Less requirements* Does not require that the switching signal is known by the decoder

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Lyapunov exponent of switched linear systems

Switched linear system: $\dot{x}(t) = A_{\sigma(t)} x(t) + B_{\sigma(t)} u(t)$

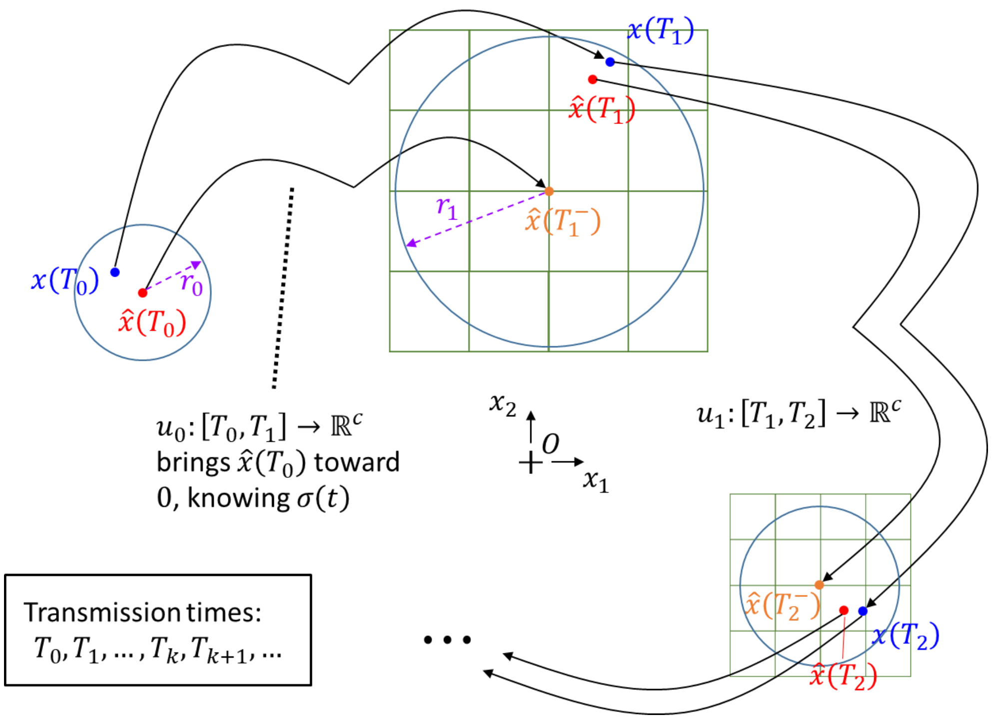
Definition: **Lyapunov exponent** of the open-loop system (i.e., with $u \equiv 0$):

$\lambda(A_\Sigma) =$ smallest α such that $\sup_{t \geq 0} e^{-\alpha t} \|x(t)\| < \infty$
for all trajectories $x(\cdot)$ of the open-loop system

= maximal "growth rate" of the trajectories of the open-loop system

- Measure of the stability of switched linear systems
- Can be approximated by numerical methods [Sun+2011, Protasov+2013]

Mode-dependent coder-controller



$$\dot{x}(t) = A_{\sigma(t)} x(t) + B_{\sigma(t)} u(t)$$

$$\dot{\hat{x}}(t) = A_{\sigma(t)} \hat{x}(t) + B_{\sigma(t)} u(t)$$

$\Downarrow$ estimation error $\delta(t) = x(t) - \hat{x}(t)$

$$\dot{\delta}(t) = A_{\sigma(t)} \delta(t)$$

… solution of the open-loop system

$$\|\delta(T_{k+1}^-)\| \leq \|\Phi(T_k, T_{k+1})\| \cdot \|\delta(T_k)\|$$

where $\Phi(s,t) = e^{A_{\sigma(t_n)}(t-t_n)} \ldots e^{A_{\sigma(t_1)}(t_1-s)}$, and $t_1, \ldots, t_n$ are the switching times in $[t, s]$

Over-approximation of the reachable set at T_{k+1}: ball centered at $\hat{x}(T_{k+1}^-)$ and with radius $r_{k+1} = \|\Phi(T_k, T_{k+1})\|\, r_k$

$\Rightarrow$ # cells in the quantizing grid $\propto \|\Phi(T_k, T_{k+1})\|^d$

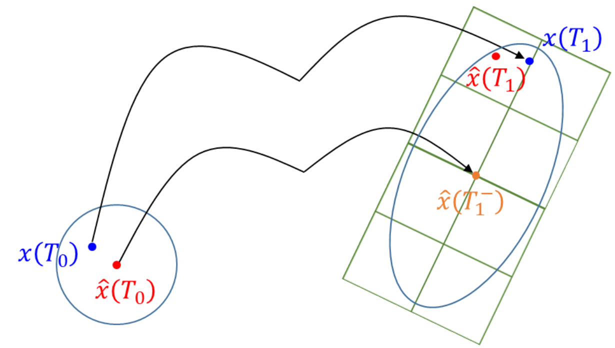
For any $\varepsilon > 0$ and τ_s large enough, $\|\Phi(T_k, T_{k+1})\| \leq e^{(\lambda(A_\Sigma)+\varepsilon)\tau_s}$

We obtain the following upper bound on the data rate:

$$R(\gamma, \zeta) \leq d \log_2(e)\, \lambda(A_\Sigma) + \varepsilon$$

Bound on the data rate

Previous scheme is **not optimal**, because the estimation error does not grow with the <u>same</u> factor in all directions!

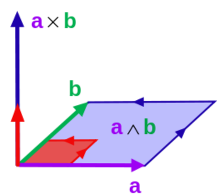


How to estimate the data rate with *adapted* grids?

Tool from multilinear algebra:

Definition: 1st-order exterior power of A is the $2^d \times 2^d$ matrix $A^{\odot}$ defined by:

$$A^{\odot} : \; w_1 \wedge \cdots \wedge w_k \; \mapsto \sum_{i=1}^{k} w_1 \wedge \cdots \wedge w_{i-1} \wedge A w_i \wedge w_{i+1} \wedge \cdots \wedge w_k$$



$w_1 \wedge \cdots \wedge w_k$ is the **wedge product** of the vectors $w_i \in \mathbb{R}^n$. It represents an element of k-volume spanned by the vectors w_i. In 3D and $k = 2$, it corresponds to the **exterior product**.

… generalization of the concept of trace of a matrix

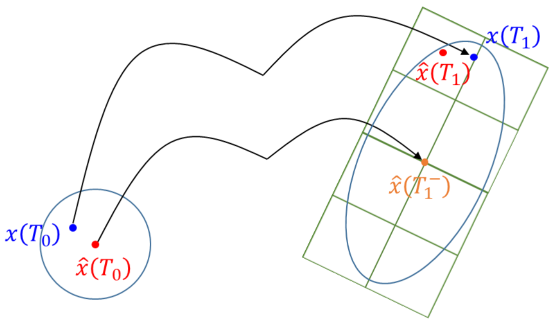
Property: $\left\| e^{A^{\odot}} \right\| = \max(\rho_1, 1) \cdots \max(\rho_n, 1), \qquad \rho_i =$ ith singular value of e^A

… recalls the well-known formula $\det(e^A) = e^{\mathrm{tr}(A)}$

Bound on the data rate

Using

- the previous scheme
- the properties of the Lyapunov exponent
- the properties of exterior power of matrices



we obtain the following **upper bound** on the data rate:

$$R(\gamma,\zeta) \leq \log_2(e)\,\lambda\left(A_\Sigma^{\odot}\right) + \varepsilon$$

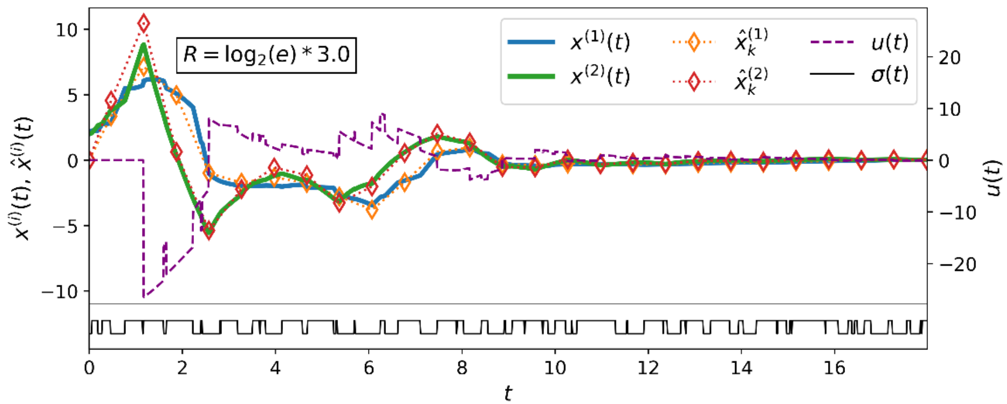
In fact, by pushing further the properties of the Lyapunov exponent and of exterior power of matrices, we can **build a switching signal for which the upper bound is tight** for all coder-controllers!

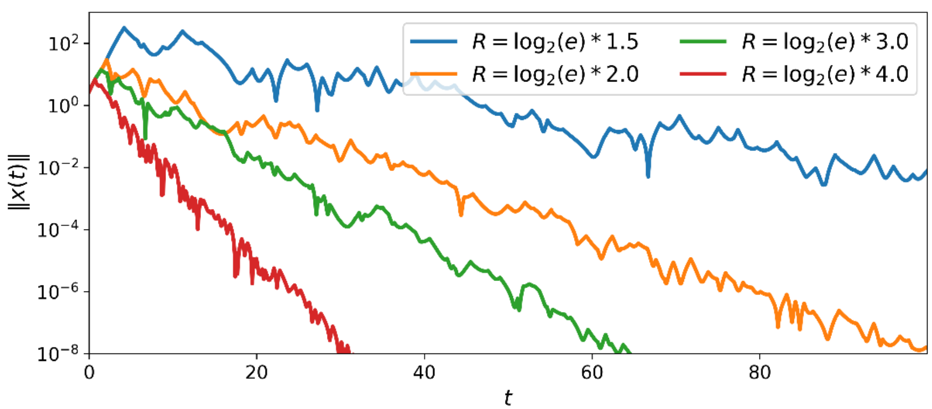
Main theorem: The **minimal data rate** R_* for stabilization of switched linear systems with a mode-dependent coder-controller is

$$R_* = \log_2(e)\,\lambda\left(A_\Sigma^{\odot}\right)$$

Numerical example

Consider the continuous-time SLS with $A_1 = \begin{bmatrix} 0.1 & 2.0 \\ 0.5 & 0.1 \end{bmatrix}$, $A_2 = \begin{bmatrix} -0.5 & 0.5 \\ 2.0 & 0.0 \end{bmatrix}$, $B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.





Exterior powers:

$$A_1^{\odot} = \begin{bmatrix} 1 & & & \\ & 0.1 & 2.0 & \\ & 0.5 & 0.1 & \\ & & & 0.2 \end{bmatrix}$$

$$A_2^{\odot} = \begin{bmatrix} 1 & & & \\ & -0.5 & 0.5 & \\ & 2.0 & 0.0 & \\ & & & -0.5 \end{bmatrix}$$

We used techniques from [Protasov+2013, Jungers+2009] to estimate $\lambda(A_\Sigma^{\odot})$: this provided

$$\boxed{1{,}21 \log_2(e) \le R_* \le 1{,}22 \log_2(e)}$$

Mode-oblivious coder-controller

Necessity of ADT > 0

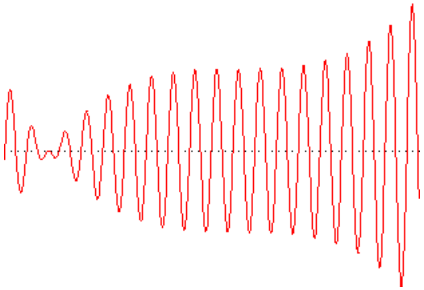
Theorem: Continuous-time switched linear systems with **zero ADT** are in general **not stabilizable** with a mode-oblivious coder-decoder with **finite data rate**

1D system: $\dot{x}(t) = B_{\sigma(t)}u(t)$ with $B_1 = -1, B_2 = +1$

… *"integrator"*: $x(T_{k+1}) - x(T_k) = \int_{T_k}^{T_{k+1}} B_{\sigma(t)}u(t)\,\mathrm{d}t$

Finite data rate ⇒ finite number of different inputs $\{u_1(\cdot), \dots, u_M(\cdot)\}$ on each transmission interval $[T_k, T_{k+1}]$

If $\sigma(\cdot)$ switches fast enough: $\left|\int_{T_k}^{T_{k+1}} B_{\sigma(t)}u_i(t)\,\mathrm{d}t\right| \leq \varepsilon$ for all i



*(formalized using the **Riemann-Lebesgue Lemma**)*

⇒ not stabilizable with finite data rate

Mode-oblivious coder-controller

Implementation and data rate

Three parameters to choose: $\tau_s > 0, n \in \mathbb{N}_{>0}, \alpha > 0$

τ_s: primary sampling period: sampling $\sigma(\cdot)$
$n\tau_s$: secondary sampling period: sampling $x(\cdot)$
α: quantization level of $x(n\tau_s)$

Result: If the parameters satisfy the following relation, then the scheme is stabilizing

$$\boxed{De^{\left(\frac{\mu_1}{\tau_a}-\mu_2\right)n\tau_s}} + e^{\frac{\mu_1}{\tau_a}n\tau_s}\boxed{e^{\nu n\tau_s}\alpha} + \frac{n\tau_s}{\tau_a}\boxed{e^{\left(\frac{\mu_1}{\tau_a}+\nu\right)n\tau_s}\tau_s D(\Delta_1 + \Delta_2 L)} < 1$$

convergence of $x(\cdot)$ without data rate constraints

effect of quantization

Frequency of unobserved switches

Deviation due to unobserved switches

$$\nu = \tfrac{1}{2}\max_{i\in\Sigma}\lambda_{\max}(A_i + A_i^\top),$$
$$\Delta_1 = \max_{i,j\in\Sigma}\|A_i - A_j\|, \quad \Delta_2 = \max_{i,j\in\Sigma}\|B_i - B_j\|.$$
$$L = \max\{\,\|\varphi(\xi,i)\| : i \in \Sigma,\ \xi \in \mathbb{R}^d,\ \|\xi\| = 1\}.$$

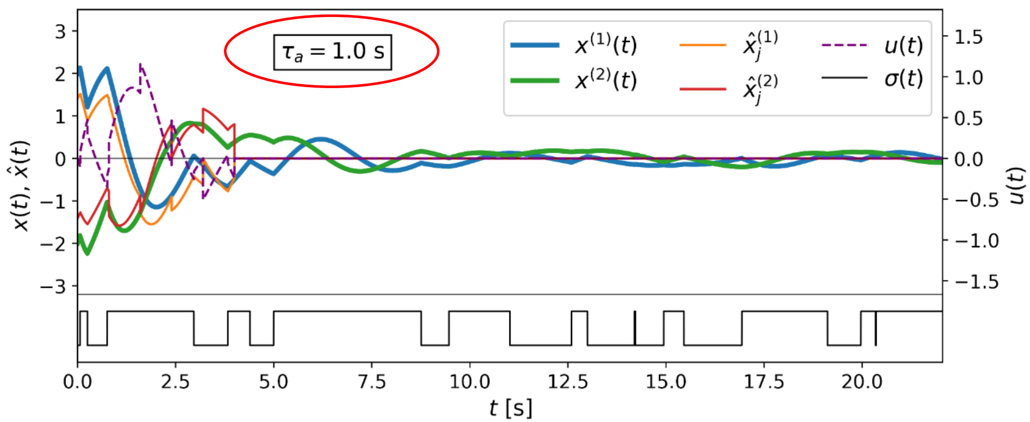
Data rate of the coder-controller:

$$R(\gamma,\zeta) = \frac{1}{\tau_s}\left[\frac{1}{n}d\log_2\left(2\left[\!\left[\frac{d^{1/2}}{2\alpha}\right]\!\right] + 1\right) + \frac{1}{n}\log_2(n+1) + \log_2|\Sigma|\right]$$

Numerical example

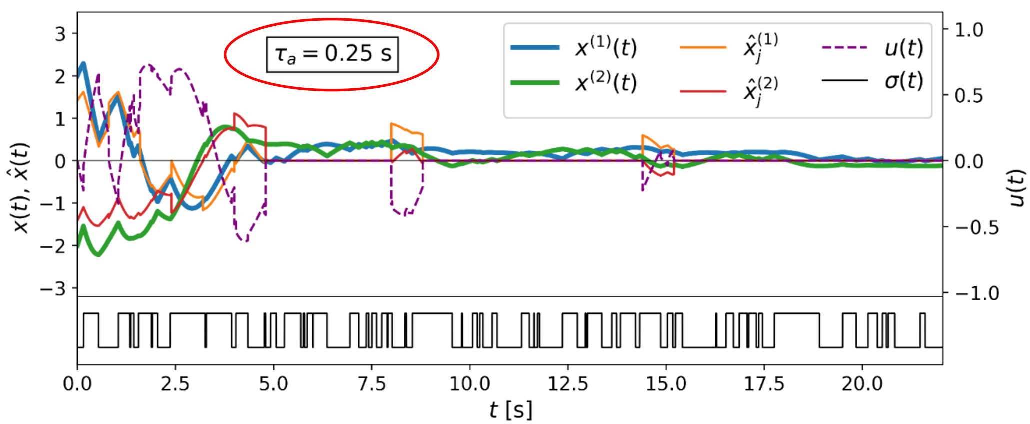
Consider the SLS (1) with matrices $A_1 = \begin{bmatrix} 0.1 & -1.0 \\ 1.5 & 0.1 \end{bmatrix}$, $A_2 = \begin{bmatrix} -0.5 & 2.0 \\ -1.5 & 0.0 \end{bmatrix}$, $B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Stabilizable with parameters
$D = 1, \mu_1 = 0, \mu_2 = 0{,}15$



$\tau_s = 0.008$, $\alpha = 0.05$ and $n = 100$

$\boxed{R = 145 \text{ bits/s}}$



$\tau_s = 0.002$, $\alpha = 0.05$ and $n = 400$

$\boxed{R = 523 \text{ bits/s}}$

Conclusions

Mode-oblivious coder-controller:

- More practical but higher data rate

Mode-dependent coder-controller:

- Lower data rate and closed-form expression for the optimal data rate
- First step toward event-based quantized control
- Fundamental lower bounds on data rate for other coders-controllers

Further works:

- Stochastic interval between switches?
- Use adapted grids for mode-oblivious coder-controller?
- Less conservative approach by uncoupling the problems of control with unknown switching signal and the problem of quantization?
- Nonlinear systems?