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St. Petersburg University

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Data rate limits for the remote state estimation

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"Remote state estimation problem: Towards the data-rate limit along the avenue of the second Lyapunov method," *Automatica*, 125, 2021

TU/e Technische Universiteit **Eindhoven** University of Technology

Outline

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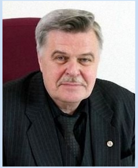
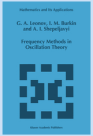
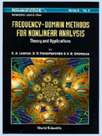
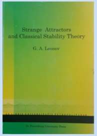
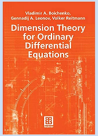


G.A. Leonov









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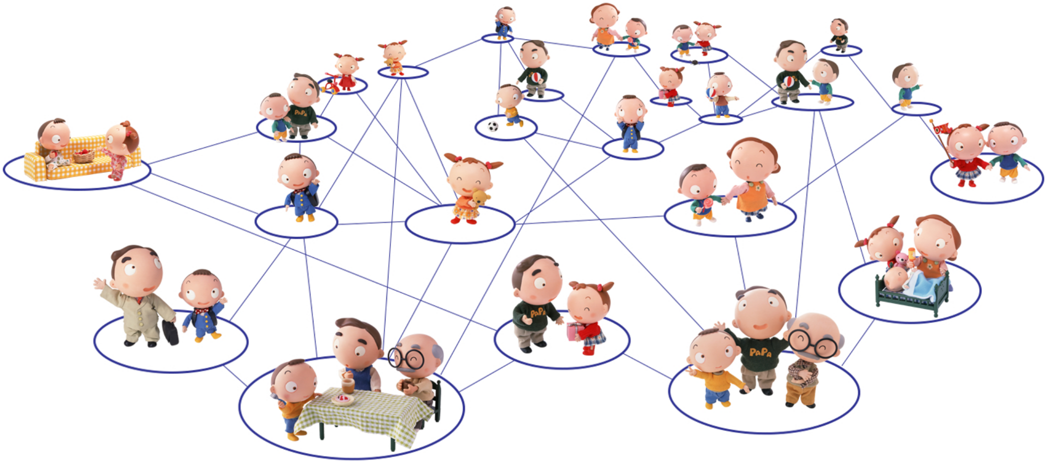
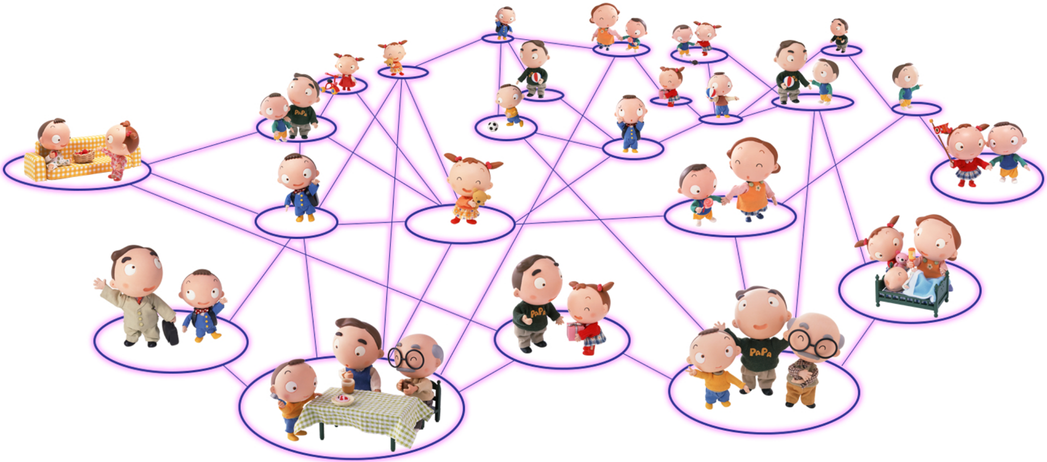
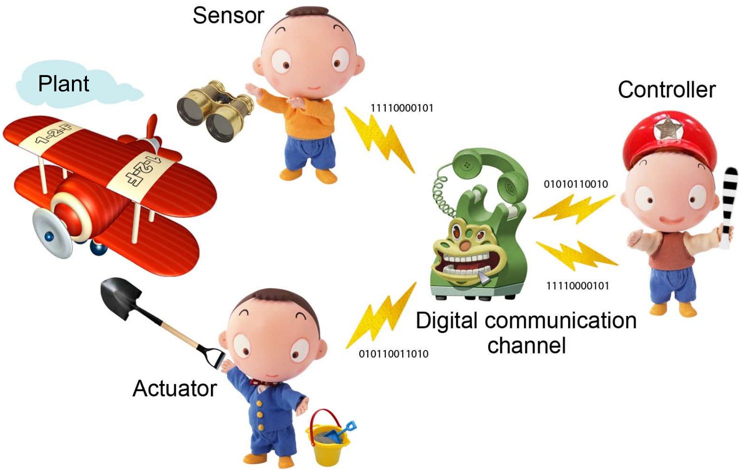
Control over communication networks



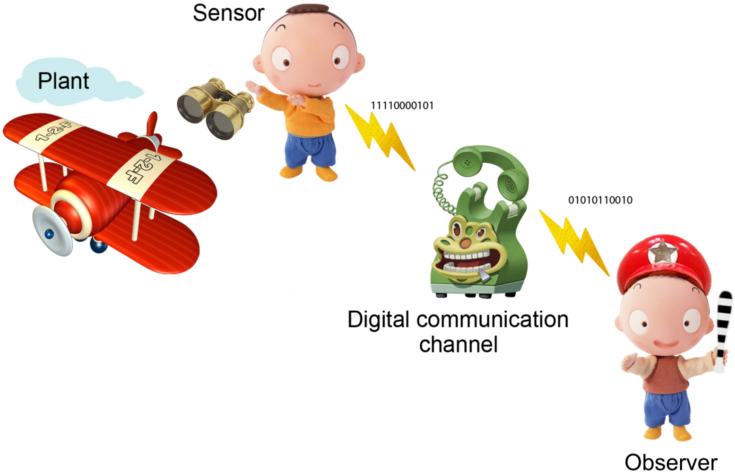
Control over communication networks



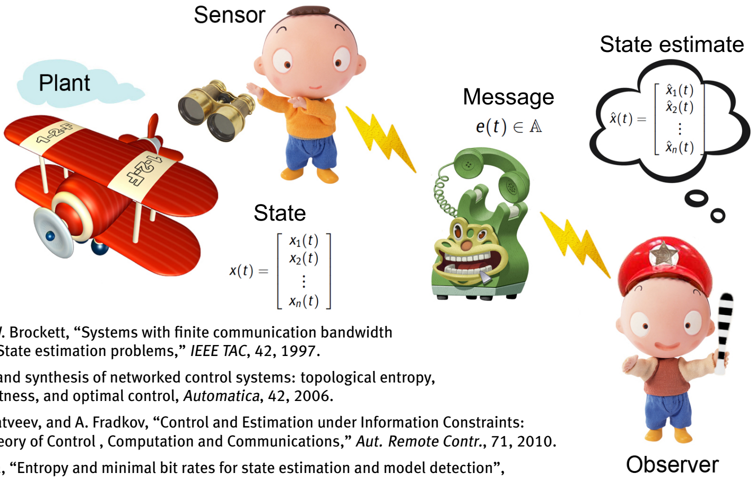
Control over communication channels



Remote state estimation problem

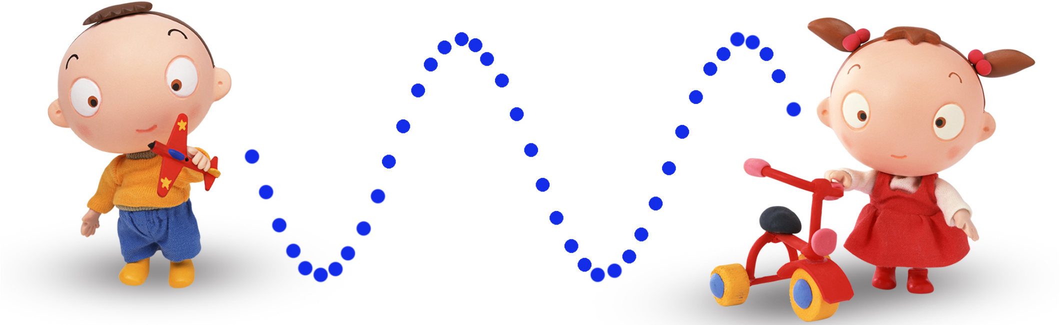


Remote state estimation problem

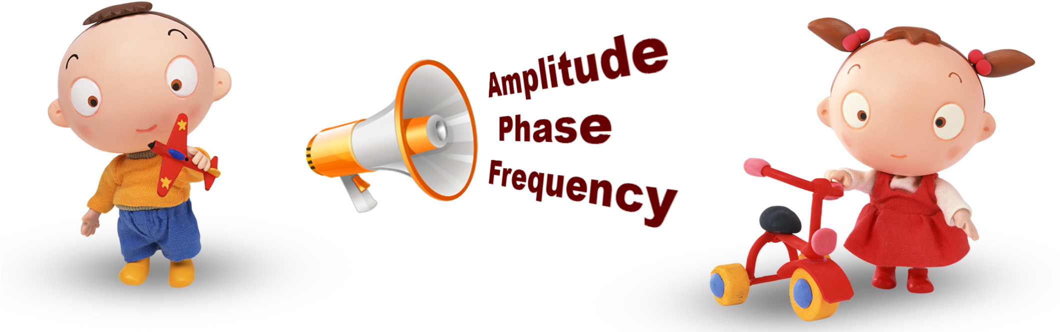


1. W. S. Wong and R. W. Brockett, "Systems with finite communication bandwidth constraints - Part I: State estimation problems," *IEEE TAC*, 42, 1997.
2. A. Savkin, Analysis and synthesis of networked control systems: topological entropy, observability, robustness, and optimal control, *Automatica*, 42, 2006.
3. B. Andrievsky, A. Matveev, and A. Fradkov, "Control and Estimation under Information Constraints: Toward a Unified Theory of Control , Computation and Communications," *Aut. Remote Contr.*, 71, 2010.
4. D. Liberzon, S. Mitra, "Entropy and minimal bit rates for state estimation and model detection", *IEEE TAC*, 63, 2018.

Remote state estimation problem



Remote state estimation problem



Remote state estimation problem

Given system, DT and CT

$$x(t+1) = \varphi(x(t)), \qquad \varphi \in C^1, \; x(0) \in K, \quad \varphi(K) \subset K, \quad K \text{ is compact}$$

Initial accuracy

$$||x(0) - \widehat{x}(0)|| \leq \delta$$

Coder

$$e(t) = \mathfrak{C}[x|_{[0,t]}, \widehat{x}(0), \delta],$$

Decoder

$$\widehat{x}(t) = \mathfrak{D}[e|_{[0,t]}, \widehat{x}(0), \delta]$$

Channel capacity

$$c = \lim_{t\to\infty} \frac{\#\text{bits}}{t},$$

Estimation goal

$$||x(t) - \widehat{x}(t)|| \quad \text{is small}$$

Remote state estimation problem

Observability

$$\forall \varepsilon > 0 \exists \delta > 0 \qquad ||x(0) - \hat{x}(0)|| \leq \delta \implies ||x(t) - \hat{x}(t)|| \leq \varepsilon, \qquad \forall t \geq 0$$

Regular observability

$$\exists \delta_*, G > 0 \ \forall \delta \leq \delta_* \qquad ||x(0) - \hat{x}(0)|| \leq \delta \implies ||x(t) - \hat{x}(t)|| \leq G\delta, \qquad \forall t \geq 0$$

Fine observability

$$\exists \delta_*, G, g > 0 \ \forall \delta \leq \delta_* \quad ||x(0) - \hat{x}(0)|| \leq \delta \implies ||x(t) - \hat{x}(t)|| \leq G\delta e^{-gt}, \qquad \forall t \geq 0$$

Remote state estimation problem

Shannon: Information is "a measure of one's freedom of choice when one selects a message". So, the source of information is the <u>uncertainty</u>.

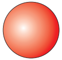
Remote state estimation problem

Alice

$$x(t+1) = \varphi(x(t))$$

$$x(t+1) = \varphi(x(t))$$

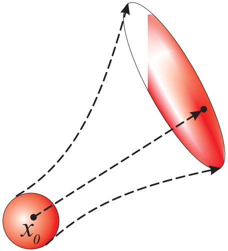
Bob



Remote state estimation problem

Alice

$x(t+1) = \varphi(x(t))$



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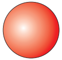

Remote state estimation problem

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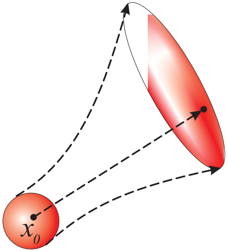

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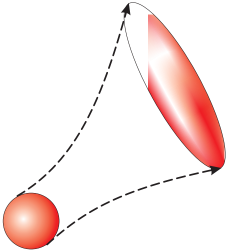

Remote state estimation problem

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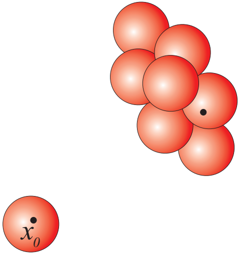

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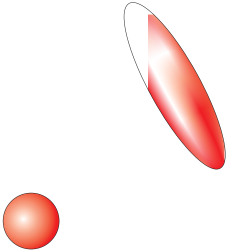

Remote state estimation problem

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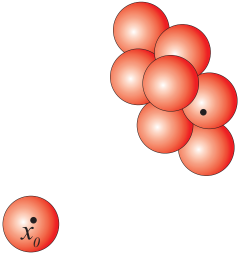

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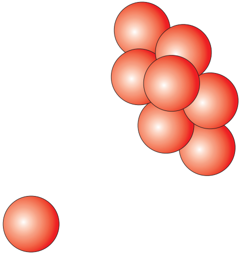

Remote state estimation problem

$x(t+1) = \varphi(x(t))$

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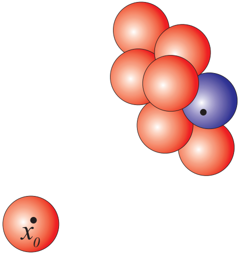

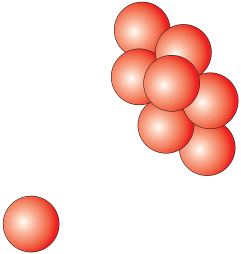

Remote state estimation problem

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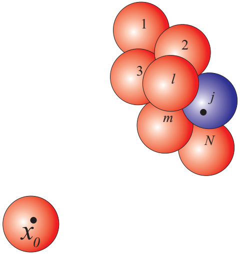
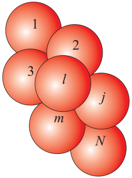
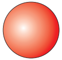
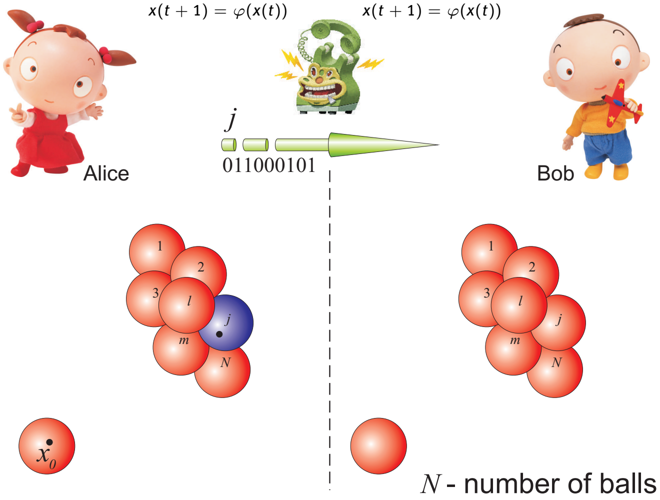




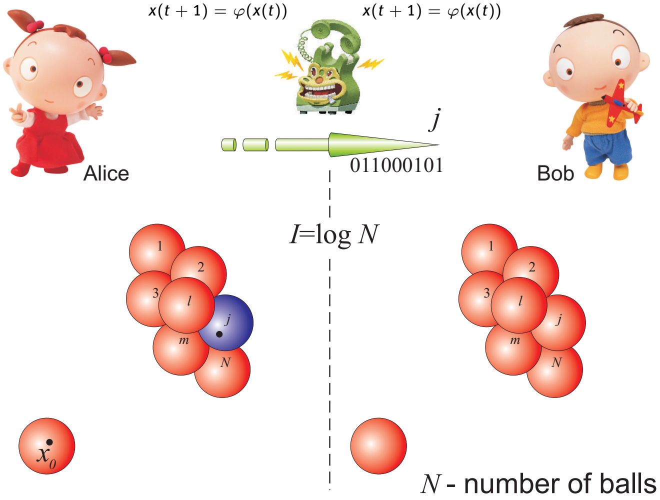


N - number of balls

Remote state estimation problem



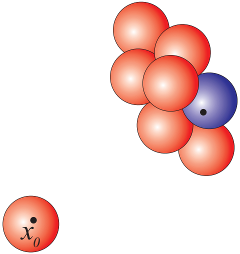
Remote state estimation problem



Remote state estimation problem

$$x(t+1) = \varphi(x(t))$$

Alice



$$x(t+1) = \varphi(x(t))$$

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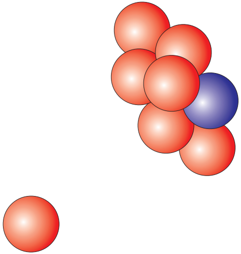

$x(t+1) = \varphi(x(t))$

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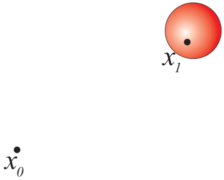
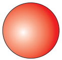




Restoration entropy

Notations

$\varphi(K) \subset K$, B_a^δ - a δ-ball centered at a. For CT $\varphi^t(\cdot)$ is the flow.
$N(T, a, \delta)$ - a minimal number of δ-balls to cover $\varphi^T(B_a^\delta \bigcap K)$.

Definition of the restoration entropy

$$\begin{aligned} H_{\text{res}}(\varphi, K) &:= \overline{\lim_{T\to\infty}} \frac{1}{T} \overline{\lim_{\delta\to 0}} \sup_{a\in K} \log_2 N(T, a, \delta) \\ &= \lim_{T\to\infty} \frac{1}{T} \overline{\lim_{\delta\to 0}} \sup_{a\in K} \log_2 N(T, a, \delta) \end{aligned}$$

Properties of the restoration entropy

Inequality, involving topological and restoration entropy

$$H_{\text{top}} \leq H_{\text{res}} \qquad \text{strict inequalities are "more often" than } =$$

1. C. Kawan, On the Relation between Topological Entropy and Restoration Entropy, *Entropy*, 21(1), 2019.
2. A. Pogromsky, A. Matveev, Data rate limitations for observability of nonlinear systems, *IFAC-PapersOnLine*, 49(14), 2016.

Data rate theorem(s)

H_{top} is a threshold of the channel capacity for observability.
H_{res} is a threshold of the channel capacity for regular(fine) observability

1. A. Savkin, Analysis and synthesis of networked control systems: topological entropy, observability, robustness, and optimal control, *Automatica*, 42, 2006.
2. A. Matveev, A. Pogromsky, Observation of nonlinear systems via finite capacity channels, Part II: Restoration entropy and its estimates, *Automatica*, 103, 2019.

Main result

$$A(x) := \mathrm{D}\varphi(x), \;\; P(x) \in C^0, \;\; P = P^\top > 0, \; \alpha_i(x) = \sqrt{\lambda_i(x)}, \;\; \det(A^\top P(\varphi(x))A - \lambda P(x)) = 0$$

Upper estimate, discrete time

$$H_{\mathrm{res}} \leq \max_{x \in K} \sum_{i=1}^{n} \max\{0, \log_2 \alpha_i(x)\}$$

Lower estimate, discrete time, (plus extra technical assumptions)

$$\forall \varepsilon > 0 \;\; \exists P(x) \in C^0, \;\; P(x) = P(x)^\top > 0, \;\; H_{\mathrm{res}} \geq \max_{x \in K} \sum_{i=1}^{n} \max\{0, \log_2 \alpha_i(x)\} - \varepsilon$$

Main result

$\dot{x} = f(x)$, $A(x) = \mathrm{D}f(x)$, σ_i are solutions of $\det(A^\top P + PA + \dot{P} - \lambda P) = 0$.

Upper estimate, continuous time

$$H_{\mathrm{res}} \leq \frac{1}{2 \ln 2} \max_{x \in K} \sum_{i=1}^{n} \max\{0, \sigma_i(x)\}$$

Lower estimate, continuous time, (plus extra technical assumptions)

$$\forall \varepsilon > 0 \ \exists P(x) \in C^1, \ P(x) = P(x)^\top > 0, \ H_{\mathrm{res}} \geq \frac{1}{2 \ln 2} \max_{x \in K} \sum_{i=1}^{n} \max\{0, \sigma_i(x)\} - \varepsilon$$

Main result

Discrete time, $x(t+1) = \varphi(x(t))$, $A := D\varphi$, $\det[A(x)^\top P(\varphi(x))A(x) - \lambda P(x)] = 0$

$$H_{\text{res}}(\varphi, K) = \inf_{P \in C^0} \max_{x \in K} \sum_{i=1}^{n} \max\{0, \log_2 \alpha_i(x)\}.$$

Continuous time, $\dot{x} = f(x)$, $A := Df$, $\det[A^\top P + PA + \dot{P} - \lambda P] = 0$

$$H_{\text{res}}(f, K) = \frac{1}{2 \ln 2} \inf_{P \in C^1} \max_{x \in K} \sum_{i=1}^{n} \max\{0, \sigma_i(x)\}$$

Example. Lanford system

$$\dot{x} = (a-1)x - y + xz$$
$$\dot{y} = x + (a-1)y + yz, \quad a > 0$$
$$\dot{z} = az - (x^2 + y^2 + z^2)$$

$$Df(x,y,z) = \begin{bmatrix} a-1+z & -1 & x \\ 1 & a-1+z & y \\ -2x & -2y & a-2z \end{bmatrix}.$$

$$P(x,y,z) = P_0 e^{w(x,y,z)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \exp\left(\frac{2z}{a}\right).$$

Restoration entropy, $a \geq 2/3$ (plus technical assumptions on K)

$$H_{\text{res}}(f,K) = \frac{2(2a-1)}{\ln 2}$$

A zoo of examples

$\dot{x} = -\sigma x + \sigma y$ $\dot{y} = rx - y - xz$ $\dot{z} = -bz + xy$	$\dot{x} = -y - z$ $\dot{y} = x$ $\dot{z} = -bz + a(y - y^2)$	$\dot{x} = -x - 4y$ $\dot{y} = x + z^2$ $\dot{z} = 1 + x$
Lorenz system	Rössler '79 system	Sprott S-system
$\dot{x} = -z$ $\dot{y} = x - y$ $\dot{z} = 3.1x + y^2 + 0.5z$	$\dot{x} = y - \mu x$ $\dot{y} = z - \nu x$ $\dot{z} = qx^2 - \gamma x$	$x(k+1) = 4x(k)(1 - x(k))$
Sprott Q-system	Ermentrout system	Logistic map
$x(k+1) = x(k) + y(k)$ $y(k+1) = \alpha y(k) - \beta \cos(x(k) + y(k))$	$x(k+1) = a + by(k) - x^2(k)$ $y(k+1) = x(k)$	$x(k+1) = 1 - a\lvert x(k)\rvert + by(k)$ $y(k+1) = x(k)$
Bouncing ball	Hénon map	Lozi map

Related results

1. A. Matveev, A. Proskurnikov, A. Pogromsky, E. Fridman, “Comprehending complexity: communication constraints in large-scale networks”, *IEEE TAC*, 64, 2019.

   A small gain theorem-like approach to estimate entropy of large-scale networks.

Related results

2. Q. Voortman, A. Pogromsky, A. Matveev, H. Nijmeijer, “Data-rate constrained observers of nonlinear systems,”, *Entropy*, 21, 2019.

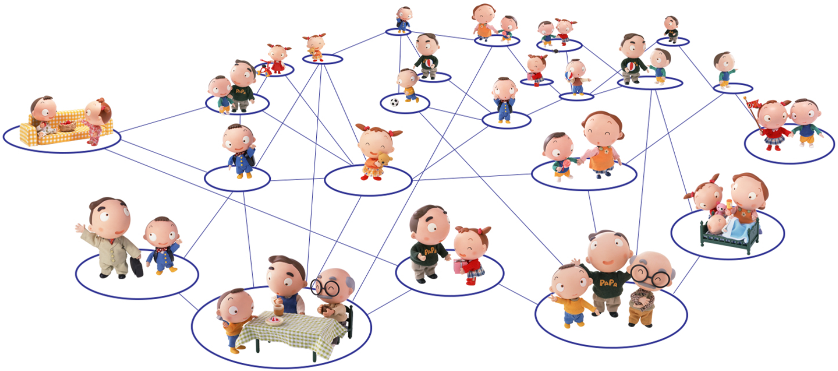
   Communication with dropouts.

Related results

3. S. Hafstein, C. Kawan, “Numerical approximation of the data-rate limit for state estimation under communication constraints, ” *Journal of Mathematical Analysis and Applications*, 473, 2019

Related results

4. Q. Voortman, A. Pogromsky, A. Matveev, H. Nijmeijer, “Data-Rate Constrained Consensus in Networks of Dynamical Systems”, in preparation.



Conclusions

- The restoration entropy provides a threshold on (regular) observability data rate for the remote state estimation problem.
- The restoration entropy is upper estimated via the singular values of $\mathrm{D}\varphi(x)$ wrt some metric $P(x)$.
- The choice of metric $P(x)$ allows to find a lower estimate as tight as one wishes.
- A number of examples with tight estimates.

Acknowledgements

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