Stochastic Stability of Non-Markovian Processes and Adaptive Quantizers

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***Abstract*— In many applications, the common assumption that a driving noise process affecting a system is independent or Markovian may not be realistic, but the noise process may be assumed to be stationary. To study such problems, this paper investigates stochastic stability properties of a class of non-Markovian processes, where the existence of a stationary measure, asymptotic mean stationarity and ergodicity conditions are studied. Applications in adaptive quantization and stochastic networked control are presented.**

I. Introduction

Consider a stationary stochastic process $\{X_k, k \in \mathbb{Z}_+\}$ where each element X_k takes values in some source space $\mathbb{X}$ (which we take to be $\mathbb{R}^n$ for some $n \in \mathbb{N}$ or some countable set) with process measure μ, and a time-invariant update rule described by

$$S_{k+1} = F(X_k, S_k) \tag{1}$$

where S_k is an $\mathbb{S}$-valued state sequence (where we take $\mathbb{S}$ also to be $\mathbb{R}^n$ for some $n \in \mathbb{N}$ or some countable subset of $\mathbb{R}^n$), with $S_0 = s$ or $S_0 \sim \kappa$ for some probability measure κ, independent of X_k. The question that we are interested in is whether for a given measurable and bounded f,

$$\lim_{N\to\infty} \frac{1}{N} E[\sum_{k=0}^{N-1} f(X_k, S_k)] \tag{2}$$

or almost surely

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} f(X_k, S_k) \tag{3}$$

exist and whether the limit is indifferent to the initial states or distributions. Here, we use the notation that capital letters denote a random variable and small letters denote the realizations. We also have $y_{[m,n]} := \{y_k, m \leq k \leq n\}$.

In (1) if $\{X_k\}$ were i.i.d, the process $\{S_k\}$ would be Markovian or if $\{X_k\}$ were Markovian, the joint process $\{(X_k, S_k)\}$ would be Markovian. For such Markov sources, there is an almost complete theory of the verification of stochastic stability through the analysis of finite-mean recurrence times to suitably defined sets [1] [16] [14] as well as the regularity properties of the kernel (such as utilizing continuity of the transition kernel and majorization by a finite measure) [14] [9]. For systems of the form (1) with only stationary $\{X_k\}$, however, there does not exist a complete theory.

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Such problems arise in many applications in feedback quantization and source coding, networks, and stochastic control. As an example, consider the following scheme which includes the Δ-Modulation [3] algorithm commonly used in source coding as a special case: Let $\{X_k\}$ be stationary and ergodic, $Q : \mathbb{R} \to \mathbb{M} \subset \mathbb{R}, |\mathbb{M}| < \infty$ be a quantizer, and consider the following update:

$$S_{k+1} = S_k + Q(X_k - S_k), \tag{4}$$

where $S_0 = 0$. Here, S_k is the output of an adaptive encoder and X_k is the source to be encoded.

The contributions in Kieffer [10] [12] [11] are the most relevant ones to the discussion in this paper. These have studied problems motivated from applications in source coding and quantization as in (4). [12] considered a non-Markovian setup where $\mathbb{S}$ is countable, [10] considered a setup where $\mathbb{S}$ is not countable, but $f(x, \cdot)$ is continuous on $\mathbb{S}^{\mathbb{N}}$ for every x. Our approach here is different than that considered in the literature; notably from that of Kieffer [10], and Kieffer and Dunham [12] (as well as other contributions such as Gersho's [3] [4] as well as [19] and [15] which can be approached by finite dimensional Markov chain formulations). We will view $(X_{(-\infty,k]}, S_k)$ or $(X_{[k,\infty)}, S_{[k,\infty)})$ as an infinite dimensional Markov chain taking values in a product space. The approach of viewing $(X_{(-\infty,k]}, S_k)$ as a Markov chain, to our knowledge, first has been studied in [7], where the focus of the author has been on the uniqueness of an invariant measure on the state process S_k, under the assumption that an invariant measure exists and further regularity assumptions. In this paper, we provide sufficient conditions for the existence of an invariant probability measure for the joint process while deriving our results. We also establish connections with asymptotic mean stationarity, in addition to the existence of an invariant measure, and ergodicity.

We will see that conditions of the form:

$$\lim_{M\to\infty} \left(\limsup_{T\to\infty} \frac{1}{T} \sum_{k=0}^{T-1} P(|S_k| \geq M) \right) = 0, \tag{5}$$

play an important role for stochastic stability.

A further related view to approach such problems is the traditional random dynamical systems view in which one studies the properties of the shifted sequences $(S_{[k,\infty)}, X_{[k,\infty)})$: Such a viewpoint leads to the interpretation that the entire uncertainty is realized in the initial state of the Markov chain, and the process evolves deterministically through a shift map. This approach has led to important contributions on ergodic theory

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and the introduction of useful notions such as asymptotic mean stationarity [5].

The proofs of the results presented here are available in [20].

II. Stochastic stability of non-Markovian systems

Towards obtaining a method to study such systems, we will here view the process $(X_{(-\infty,k]}, S_k)$ as a $\mathbb{X}^{\mathbb{Z}_-} \times \mathbb{S}$-valued Markov process. We recall that with $\mathbb{X}$ a complete, separable, metric (that is, a Polish) space, $\Sigma = \mathbb{X}^{\mathbb{Z}_-}$ is also a Polish space under the product topology. By a standard argument (e.g. *Chapter* 7 in [2]), we can embed the one-sided stationary process $\{X_k, k \in \mathbb{Z}_+\}$ into a bilateral (double-sided) stationary process $\{X_k, k \in \mathbb{Z}\}$. We first state the following.

Lemma II.1. *The sequence (Z_k, S_k) with $Z_k = X_{(-\infty,k]}$ is a Markov process.*

We let $\mathbb{P}$ denote the transition kernel for this process. Since X_k is known to be stationary, if there were an invariant measure v for this process, then this would decompose as

$$v(ds_0|x_{(-\infty,0]})\pi(dx_{(-\infty,0]})$$

with π being the stationary measure for X_k.

If there is an invariant probability measure $\bar{P}$ for such a process we say that the process is *stochastically stable*. By the ergodic theorem [9, Theorems 2.3.4-2.3.5], $\bar{P}$ almost surely

$$\lim_{N\to\infty} \frac{1}{N} E_{x_{(-\infty,0]},s}[\sum_{k=0}^{N-1} f(X_{(-\infty,k]}, S_k)] = f^*(x_{(-\infty,0]}, s) \tag{6}$$

exists for all measurable and bounded f and for corresponding functions f^* (where the full set of convergence may depend on the function f). The following assumption will be useful in establishing further stability results in Section III. Recall that $S_0 \sim \kappa$ for some probability measure κ.

Assumption II.1. *The invariant measure $\bar{P}$ is such that $\pi \times \kappa \ll \bar{P}$. That is, $\bar{P}(A, B) = 0$ implies that $\pi(A)\kappa(B) = 0$ for any Borel A, B.*

Under this assumption, the set of initial conditions which may not satisfy (6) (this set has zero measure under $\bar{P}$) also has zero measure under the initial product probability measure $\pi \times \kappa$. Furthermore, sample paths also converge almost surely

A. Existence of an invariant probability measure with finite $\mathbb{S}$

Our first result is for the setup with finite $\mathbb{S}$. For some related results and an alternative approach for the finite case, see [13].

Theorem II.1. *Consider the dynamical system given by (1). Suppose that $\mathbb{S}$ is finite. Then, the process is stochastically stable.*

Proof sketch. Define for all $a \in \mathbb{S}$, the sequence of expected occupational measures

$$v_t(dx_{(-\infty,k]} \times \{a\}) = E[\frac{1}{t}\sum_{k=0}^{t-1} 1_{\{X_{(-\infty,k]}, S \in dx_{(-\infty,k]} \times \{a\}\}}]$$

$$= \frac{1}{t}\sum_{k=0}^{t-1} P(X_{(-\infty,k]}, S \in dx_{(-\infty,k]} \times \{a\}), \tag{7}$$

where for every k, $X_{(-\infty,k]} \sim \pi$. Therefore, for any t, we can write

$$v_t(dx_{(-\infty,k]} \times \{a\}) = \pi(dx_{(-\infty,k]})v_t(a|x_{(-\infty,k]})$$

since π is a stationary measure. It follows then that

$$v_t(dx_{(-\infty,k]} \times \{a\}) \leq \pi(dx_{(-\infty,k]})$$

for every a and $|\mathbb{S}|\pi(dx_{(-\infty,k]})$ is a majorizing *finite* measure for the sequence v_t. This ensures by [9, Proposition 1.4.4] that the sequence $\{v_t\}$ has a converging subsequence v_{t_k} in the setwise sense so that for some probability measure v, $v_{t_k}(A) \to v(A)$ for all Borel A. Some further analysis reveals that v is an invariant probability measure. □

B. Existence of an invariant probability measure with countable $\mathbb{S}$

Here, we assume that $\mathbb{S}$ is a countable set viewed as a subset of $\mathbb{R}$ whose elements are uniformly separated from each other; thus $\mathbb{S}$ is a uniformly discrete set in the sense that there exists $r > 0$ such that $|x - y| > r$ for all $x, y \in \mathbb{S}$.

Theorem II.2. *Consider the dynamical system given by (1). If (5) holds with the norm defined on $\mathbb{R}$, the process is stochastically stable.*

C. Existence of an invariant probability measure with $\mathbb{S} = \mathbb{R}^n$

We have the following assumption.

Assumption II.2. *$F(x, s)$ is continuous in s for every x.*

Theorem II.3. *Consider the dynamical system given by (1). If (5) holds, under Assumption II.2, the process is stochastically stable.*

A large class of applications do not have the property that $\mathbb{S}$ is countable or that F is continuous in s. To approach such problems in our framework, we impose the following quasi-Feller type condition which is natural for the applications we will consider.

Assumption II.3. *$F(x, s)$ is continuous on $\mathbb{X}\times\mathbb{S}\setminus D$ where D is a closed set with $P((X_{t+1}, S_{t+1}) \in D|x_{(-\infty,t]} = x, s_t = s) = 0$ for all x, s. Furthermore, with $D_\epsilon = \{z : d(z, D) < \epsilon\}$ for $\epsilon > 0$ and d the product metric on $\mathbb{X}\times\mathbb{S}$, for some $K < \infty$, we have that for all x, s and $\epsilon > 0$*

$$P\bigg((X_{t+1}, S_{t+1}) \in D_\epsilon|x_{(-\infty,t]} = x, s_t = s\bigg) \leq K\epsilon.$$

Assumption II.4. (i) *If $\mathbb{X}$ is compact, for every continuous and bounded f, $\int_{\mathbb{X}} P(X_{k+1} \in dx|X_{(-\infty,k]} = z)f(x)$ is continuous in z.*

(ii) *If $\mathbb{X}$ is not compact, $\int_{\mathbb{X}} P(X_{k+1} \in dx|X_{(-\infty,k]} = z)f(x)$ is continuous in z for every measurable and bounded f.*

Theorem II.4. *Suppose that Assumptions II.3 and II.4 hold. If (5) holds, the system is stochastically stable.*

III. Asymptotic mean stationarity and ergodicity

A. Shifts and random dynamical systems view

Let $\mathbb{X}$ be a complete, separable, metric space. Let $\mathcal{B}(\mathbb{X})$ denote the Borel sigma-field of subsets of $\mathbb{X}$, let $\Sigma = \mathbb{X}^{\mathbb{Z}_+}$ denote the sequence space of all one-sided (unilateral) infinite sequences drawn from $\mathbb{X}$. Thus, if $x \in \Sigma$ then $x = \{x_0, x_1, x_2, \dots\}$ with $x_i \in \mathbb{X}$. Let $X_n : \Sigma \to \mathbb{X}$ denote the coordinate function such that $X_n(x) = x_n$. Let T denote the shift operation on Σ, that is $X_n(Tx) = x_{n+1}$. With $\mathbb{X}$ a Polish space, $\Sigma = \mathbb{X}^{\mathbb{Z}_+}$ is also a Polish space under the product topology. Let $\mathcal{B}(\Sigma)$ denote the smallest σ-field containing all cylinder sets of the form $\{x : x_i \in B_i, m \le i \le n\}$ where $B_i \in \mathcal{B}(\mathbb{X})$, for all integers $m, n \ge 0$. Here, $\cap_{n\ge0} T^{-n}\mathcal{B}(\Sigma)$ is the tail $\sigma-$field: $\cap_{n\ge0}\sigma(X_n, X_{n+1}, \cdots)$, since $T^{-n}(A) = \{x : T^n x \in A\}$.

Let μ be the measure on the process $\{X_0, X_1, \cdots\}$. This process is stationary and μ is said to be a stationary (or invariant) measure on $(\Sigma, \mathcal{B}(\Sigma))$ if $\mu(T^{-1}B) = \mu(B)$ for all $B \in \mathcal{B}(\Sigma)$. This random process is ergodic if $A = T^{-1}A$ implies that $\mu(A) \in \{0, 1\}$.

Definition III.1. *[6] A process on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with process measure μ, is asymptotically mean stationary (AMS) if there exists a probability measure $\bar{P}$ such that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \mu(T^{-k}F) = \bar{P}(F), \tag{8}$$

for all events $F \in \mathcal{B}(\Sigma)$. Here $\bar{P}$ is called the stationary mean of μ, and is a stationary measure.

$\bar{P}$ is stationary since, by definition $\bar{P}(F) = \bar{P}(T^{-1}F)$. Due to the Markov formulation, we can obtain the following to check whether the AMS property holds under (1).

Theorem III.1. *Let there exist a stationary measure $\bar{P}$ for the Markov chain $(X_{(-\infty,k]}, S_k)$ for the system (1). Assumption II.1 implies the AMS property for the process (X_k, S_k).*

B. Ergodicity

For a Markov chain, the uniqueness of an invariant probability measure implies ergodicity (see e.g. [9, Chp. 2]). Consider an $\mathbb{X}$-valued Markov chain with transition kernel P, where $\mathbb{X}$ is a Polish space.

Definition III.2. *A Markov chain is μ-irreducible, if for any set $B \in \mathcal{B}(\mathbb{X})$ such that $\mu(B) > 0$, and $\forall x \in \mathbb{X}$, there exists some integer $n > 0$, possibly depending on B and x, such that $P^n(x, B) > 0$, where $P^n(x, B)$ is the transition probability in n stages from x to B.*

A maximal irreducibility measure ψ is an irreducibility measure such that for all other irreducibility measures ϕ, we have $\psi(B) = 0 \Rightarrow \phi(B) = 0$ for any $B \in \mathcal{B}(\mathcal{X})$.

Theorem III.2. *For a ψ-irreducible Markov chain, there can be at most one invariant probability measure.*

A complementary condition for ergodicity is the following.

Definition III.3. *For a Markov chain with transition kernel P, a point x is accessible if for every y and every open neighbourhood O of x, there exists $k > 0$ such that $P^k(y, O) > 0$.*

One can show that if a point is accessible, it belongs to the (topological) support of every invariant measure (see, e.g., Lemma 2.2 in [8]). Recall that the support of a probability measure is defined to be the set of all points x for which every open neighbourhood of x has positive measure.

We recall that a Markov chain V_t is said to have the strong Feller property if $E[f(V_{t+1})|V_t = v]$ is continuous in v for every measurable and bounded f.

Theorem III.3. *[8] [17] If a Markov chain over a Polish space has the strong Feller property, and if there exists an accessible point, then the chain can have at most one invariant probability measure.*

However, a Markov chain defined as $(X_{(-\infty,k]}, S_k)$ cannot be strongly Feller due to the memory in the source.

Theorem III.4. *Suppose that $E[f(X_{k+1}, S_{k+1})|X_{(-\infty,k]} = x_{(-\infty,k]}, S_k = s_k]$, for measurable and bounded $f : \mathbb{X} \times \mathbb{S} \to \mathbb{R}$, is continuous in $(x_{(-\infty,k]}, s_k)$. Suppose further that there exists an accessible point for the Markov chain $\{(X_{(-\infty,k]}, S_k)\}$. The chain can have at most one invariant probability measure.*

For applications such as Δ-Modulation, however, we will see that the continuity assumption in Theorem III.4 fails to hold. To be able to apply the result for such setups, we have the following relaxation.

Theorem III.5. *Suppose that for measurable and bounded $f : \mathbb{X} \times \mathbb{S} \to \mathbb{R}$, $E[f(X_{k+1}, S_{k+1})|X_{(-\infty,k]} = x_{(-\infty,k]}, S_k = s_k]$ is continuous in $x_{(\infty,k]}, s_k$ for all $x, s \in (\mathbb{X}^{\mathbb{Z}_-} \times \mathbb{S}) \setminus D$ for some closed set D. Suppose further that there exists an accessible point $(x, s) \notin D$ for the Markov chain $\{(X_{(-\infty,k]}, S_k)\}$. The chain can have at most one invariant probability measure.*

IV. Applications to Stochastic Stability of Adaptive Quantizers

In this section, we consider applications in feedback quantization and networked control.

A. Adaptive Quantization with a Stationary Source

Adaptive quantization for stationary sources has been studied in particular in [10], [12] and [3].

1) Δ-Modulation:

Theorem IV.1. *Let X_k be stationary and ergodic $\mathbb{R}$-valued process stationary process measure π, $Q : \mathbb{R} \to \{-m, m\}$, with the following update:*

$$S_{k+1} = S_k + Q(X_k - S_k),$$

where $S_0 = 0$ and $Q(Z) = m1_{\{Z\ge0\}} - m1_{\{Z<0\}}$. Suppose further that $E[Q(X_0-m)] < 0$ and $E[Q(X_0+m)] > 0$ (equivalently $P(X_0 \ge m) < 1/2, P(X_0 \le -m) < 1/2$). Then, the system is stochastically stable in the sense that there exists an invariant probability measure. Furthermore, if for every m, k, and non-empty open A_k, $\pi(X_{[m,k]} \in \prod_{t=m}^{k} A_k) > 0$, the *system is AMS. If in addition $E[g(X_1)|x_{(-\infty,0]}]$ is continuous in $x_{(-\infty,0]}$ for measurable and bounded g, (X_k, S_k) is ergodic.*

Here $\mathbb{S} = \{km, k \in \mathbb{Z}\}$ is a countable set. An example where the condition $(\pi(X_{[m,k]} \in \prod_{t=m}^{k} A_k) > 0)$ holds for non-empty open sets is $X_{t+1} = \sum_{i=0}^{\infty} \alpha_i W_{t-i}$, with $\sum_t |\alpha_t^2| < \infty$ and W_t is a sequence of i.i.d. Gaussian random variables. An example where the continuity condition for ergodicity holds is the following auto-regressive representation $X_{t+1} = \sum_{i=0}^{N-1} \alpha_i X_{t-i} + W_t$, with the roots of $1 - \sum_{i=1}^{N} \alpha_{i-1} z^{-i}$ strictly inside the unit circle and W_t a sequence of i.i.d. Gaussian random variables.

Proof sketch. It can be shown that (5) can be equivalently written as $\lim_{M\to\infty} \left(\liminf_{T\to\infty} \frac{1}{T} \sum_{k=0}^{T-1} P(|S_k| < M) \right) = 1$. This is implied by [12, Eqn. (2.2)], as a result of [12, Theorem 2]. This follows since, with K a finite set, the condition $S_i \in K$ for some $i \in \{n, \cdots, n+N\}$ [12, Eqn. (2.2)] implies that $|S_i| \leq K_1 + Nm \leq K_2 N$ for constants K_1, K_2 since $|S_i - S_j| \leq |i-j|m$. By Theorem II.3, the system is stochastically stable.

Asymptotic mean stationarity: For the AMS property, we show that Assumption II.1 holds: Let $X_{[-m,0]} \in B$ and $S_0 = 0$ have a zero measure under $\bar{P}$. Then, $\pi(B) = 0$. To show this, consider the contrapositive: If $\pi(B) > 0$, by the condition that all finite-dimensional cylinder sets consisting of non-empty open sets have positive measure conditioned on any past event, it follows that for some $S_0 = s^*$ with positive measure under $\bar{P}$, there exists a positive probability event $X_{[0,m]} \in B$ so that $S_m = 0$. With,

$$\bar{P}(X_{[-m,0]} \in B, S_0 = 0) = \bar{P}(X_{[0,m]} \in B, S_m = 0)$$
$$\geq \int_z \bar{P}(dz, s^*) \mathbb{P}(x_{[0,m]} \in B, S_m = 0 | z, s^*) > 0, \quad (9)$$

it follows that the absolute continuity condition holds, and by Theorem III.1, the AMS property.

Ergodicity: We can establish the uniqueness of an invariant probability measure through either irreducibility properties or the following argument. Consider the point $p_0 = \{m/2\}^{\mathbb{Z}_-} \times \{0\}$. We argue that this point is accessible. Recall that an open set in a product topology is a Borel set in the product space consisting of finitely many open sets with the rest being $\mathbb{X}$ itself or arbitrary union of such sets. Now, consider any $x_{(-\infty,0]}, s$. From this point, we can show that for every open neighborhood U of p_0, there exists some $k > 0$ so that $P(X_{(-\infty,k]} \in U | x_{(-\infty,0]}, s) > 0$. Also the sets of points where continuity fails, $D = \{x : x = km, k \in \mathbb{Z}\}$, is a closed set and p_0 is outside this set. By Theorem III.5, the process is ergodic. □

Remark IV.1. *Stochastic stability can also be established for leaky adaptive quantization [3] with $S_{k+1} = \alpha S_k + Q(X_k - S_k)$ for some $|\alpha| \leq 1$. Here, since the set of values that S can take can be a continuum, Theorem II.4 needs to be invoked to ensure that an invariant probability measure exists.*

A final remark is that the Δ-modulation leads to a process that is periodic since the state process for the quantizer (shifted by the initial condition) cannot take consecutive even or odd values. This is unlike the setup that we will consider in the application from networked control further below.

2) Adaptive Quantization of Goodman and Gersho: Consider the following update equations [4]:

$$V_t = \Delta_t Q_1(X_t/\Delta_t)$$
$$\Delta_{t+1} = \Delta_t Q_2(\frac{|X_t|}{\Delta_t}), \quad \Delta_0 = b \quad (10)$$

Here, Δ_t is the bin size of the uniform quantizer with a finite range and $|Q_1(\mathbb{R})| < \infty, |Q_2(\mathbb{R}_+)| < \infty$. V_t is the output which is to *track* the source process X_t. Suppose further that Q_2 is non-decreasing.

Theorem IV.2. *Let X_t be a stationary and ergodic Gaussian sequence, $\zeta = \lim_{x\to\infty} Q_2(x) > 1$, $Q_2(0) = \lim_{x\downarrow 0} Q_2(x) < 1$ and $\log_2(Q_2(\cdot)) \in \mathbb{Q}$. Then, the system is stochastically stable. If in addition, with $\{\alpha_1, \alpha_2, \cdots, \alpha_L\}$ a set of pairwise relatively prime integers and $\log_2(Q_2(\cdot)) \in \{\alpha_k m\}$ for some $m \in \mathbb{Q}$, the process is AMS, and furthermore, ergodic.*

Proof sketch. Consider

$$\log_2(\Delta_{t+1}) = \log_2(\Delta_t) + \log_2(Q_2(\frac{|X_t|}{\Delta_t}))$$

$\log_2(\Delta_t) - \log_2(\Delta_0) \in \mathbb{Q}$ for all t. Let $S_t = \log_2(\Delta_t)$. This sequence takes values in a countable set and satisfies

$$S_{t+n} - S_t = \sum_{k=t}^{t+n-1} \log_2(Q_2(\frac{|X_k|}{\Delta_k})).$$

As in the proof of Theorem IV.1, by Theorem II.3 the system is stochastically stable.

The AMS property: Since $\{\alpha_k\}$ is a set of numbers that are relatively prime, $\mathbb{S}$ consists of all integer multiples of m shifted by the initial value $\log_2(b)$. This follows from the property of relatively prime numbers due to Bézout's lemma; see [21, Lemma 7.6.2]. The argument for the AMS property then follows as before through the absolute continuity condition: Any invariant measure is such that $\bar{P}(\cdot, s) \ll \bar{P}(\cdot, s')$ for all admissible s, s' and by Theorem III.1, the result follows.

Ergodicity: In this case, the point $(\{0\}^{\mathbb{Z}_-}, \log_2(b))$ is accessible by the same arguments adopted in the proof of Theorem IV.1 and the steps leading to the AMS property above. By Theorem III.5, the process is ergodic. □

B. Stochastic networked control

We consider a stabilization problem in stochastic networked control where a linear system is controlled over a communication channel. We will study the approach in [19], [22] (see [21] for a detailed discussion). Consider the following control system, with U_t a control variable,

$$X_{t+1} = aX_t + bU_t + W_t. \quad (11)$$

where $|a| \geq 1$, W_t is i.i.d, admitting a probability measure v which admits a density, positive everywhere and bounded.

Furthermore, $E[|W_t|^{2+\zeta}] < \infty$ for some $\zeta > 0$. In the application considered, a controller has access to quantized information from the state process. The quantization is described as follows. An adaptive quantizer has the following form with Q_K^{Δ} being a uniform quantizer with $K+1$ bins and bin-size Δ, $Q_K^{\Delta} : \mathbb{R} \to \mathbb{R}$ satisfies the following for $k = 1, 2 \ldots, K$:

$$Q_K^{\Delta}(x) = \begin{cases} (k - \frac{1}{2}(K+1))\Delta, \\ \qquad\qquad \text{if} \quad x \in [(k-1-\frac{1}{2}K)\Delta, (k-\frac{1}{2}K)\Delta) \\ \frac{1}{2}(K-1)\Delta, \qquad \text{if} \quad x = \frac{1}{2}K\Delta \\ 0, \qquad \text{if} \quad x \notin [-\frac{1}{2}K\Delta, \frac{1}{2}K\Delta] \end{cases}$$

With $K = \lceil |a| + \epsilon \rceil$, $R = \log_2(K+1)$, let $R' = \log_2(K)$. We will consider the following coding and quantization update policy. For $t \geq 0$ and with $\Delta_0 > L$ for some $L \in \mathbb{R}_+$, and $\hat{x}_0 \in \mathbb{R}$, consider:

$$U_t = -\frac{a}{b}\hat{X}_t, \quad \hat{X}_t = Q_K^{\Delta_t}(X_t),$$
$$\Delta_{t+1} = \Delta_t \bar{Q}(|\frac{X_t}{\Delta_t 2^{R'-1}}|, \Delta_t)$$

Suppose that with $\delta, \epsilon, \alpha > 0$ with $\alpha < 1$ and $L > 0$ the following hold

$$\begin{aligned} \bar{Q}(x,\Delta) &= |a| + \delta \quad \text{if} \quad |x| > 1 \\ \bar{Q}(x,\Delta) &= \alpha \quad \text{if} \quad 0 \leq |x| \leq 1, \Delta \geq L \\ \bar{Q}(x,\Delta) &= 1 \quad \text{if} \quad 0 \leq |x| \leq 1, \Delta < L, \end{aligned}$$

Theorem IV.3. *[22] [19] Consider an adaptive quantizer applied to the linear control system described by (11). If the noiseless channel has capacity,*

$$R > \log_2(\lceil |a| \rceil + 1),$$

and for the adaptive quantizer in (12), if the quantizer bin sizes are such that their (base−2) logarithms are integer multiples of some scalar s, and $\log_2(\bar{Q}(\cdot,\cdot))$ take values in integer multiples of s where the integers taken are relatively prime (that is they share no common divisors except for 1), then the process $\{(X_t, \Delta_t)\}$ is a positive (Harris) recurrent Markov chain (and has a unique invariant distribution).

In [22] it was shown that an m-small set (since a petite set in an irreducible and aperiodic Markov chain is m-small [14]) can be constructed so that return conditions are satisfied. Hence, the return time properties directly leads to a stability result. The small set discussion in [22] builds on the Markovian property and irreducibility and aperiodicity of the Markov chain, together with a *uniform countable additivity* condition from [18]. We can obtain the stability result through the analysis in this paper, without defining a small/petite set: One can view the system as: $(\Delta_{t+1}, x_{t+1}) = F(\Delta_t, x_t, w_t)$, where the state is now $s_t := (\Delta_t, x_t)$ and the independence of w_t makes the process (Δ_t, x_t) Markov. Let the transition kernel be denoted with P. The finiteness of $\limsup_{t\to\infty} E[\Delta_t^2 + x_t^2]$ can be established by a Lyapunov analysis similar to [19] and [22]. However, F here is not continuous in s_t. Nonetheless, the set of discontinuity is given by: $D = \{x, \Delta : \frac{x}{\Delta} \in \{-\frac{K}{2}, \cdots, \frac{K}{2}\}, \quad \Delta \in \mathcal{N}\}$, where $\mathcal{N}$ is the set of admissible bin sizes which is a countable set by the hypothesis of relative primeness. As a result D is also countable and closed (since the elements are uniformly separated from each other). Furthermore, any weak limit of a converging sequence of expected occupational measures has zero measure on D, as can be deduced from the condition that every open set $D_\epsilon = \{x, \Delta : d((x,\Delta), D) < \epsilon\}$ is such that $v_{t_k} P(D_\epsilon) \leq L_1 \epsilon$, for some $L_1 < \infty$ since $P(x_{t+1} \in dx | x, \Delta)$ has a density which is uniformly bounded for all z, Δ and the conditional probability $P(\Delta_{t_k} | x_{t_k - 1} = z, \Delta_{t_k - 1} = \Delta)$ has finite support. By Theorem II.4, the result follows. Finally, ergodicity follows from the irreducibility of the Markov process. □

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