

ON STOCHASTIC STABILITY OF A CLASS OF NON-MARKOVIAN PROCESSES AND APPLICATIONS IN QUANTIZATION*

SERDAR YÜKSEL†

Abstract. In many applications, the common assumption that a driving noise process affecting a system is independent or Markovian may not be realistic, but the noise process may be assumed to be stationary. To study such problems, this paper investigates stochastic stability properties of a class of non-Markovian processes, where the existence of a stationary measure, asymptotic mean stationarity, and ergodicity conditions are studied. Applications in feedback quantization and stochastic control are presented.

Key words. stochastic stability, non-Markovian processes, ergodicity, stochastic control, feedback quantization

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1. Introduction. Consider a stationary stochastic process $\{X_k, k \in \mathbb{Z}_+\}$ where each element X_k takes values in some source space \mathbb{X} (which we take to be \mathbb{R}^n for some $n \in \mathbb{N}$ or some countable set) with process measure μ , and a time-invariant update rule described by

$$(1.1) \quad S_{k+1} = F(X_k, S_k),$$

where S_k is an \mathbb{S} -valued state sequence (where we take \mathbb{S} also to be \mathbb{R}^n for some $n \in \mathbb{N}$ or some countable subset of \mathbb{R}^n), with $S_0 = s$ or $S_0 \sim \kappa$ for some probability measure κ , independent of X_k . The question that we are interested in is whether for a given measurable and bounded f ,

$$(1.2) \quad \lim_{N \rightarrow \infty} \frac{1}{N} E \left[\sum_{k=0}^{N-1} f(X_k, S_k) \right]$$

or almost surely

$$(1.3) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} f(X_k, S_k)$$

exist and whether the limit is indifferent to the initial states/distributions. The function f can be taken to be more general as follows:

$$(1.4) \quad \lim_{N \rightarrow \infty} \frac{1}{N} E \left[\sum_{k=0}^{N-1} f(X_{[k, \infty)}, S_{[k, \infty)}) \right]$$

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†Department of Mathematics and Statistics, Queen's University, Kingston, Ontario, K7L 3N6, Canada (yuksel@mast.queensu.ca).

or

$$(1.5) \quad \lim_{N \rightarrow \infty} \frac{1}{N} E \left[\sum_{k=0}^{N-1} f(X_{(-\infty, k]}, S_k) \right].$$

Here, we use the notation that capital letters denote a random variable and small letters denote the realizations. We also have $y_{[m, n]} := \{y_k, m \leq k \leq n\}$. One may also add another variable $U_k = g(S_k, X_k)$, where U_k is an output of the system taking values in some set \mathbb{U} and revise the formulation of the problem accordingly. We note that all of the random variables are defined on a common probability space (Ω, \mathcal{F}, P) .

In (1.1) if $\{X_k\}$ were independently and identically distributed (i.i.d), the process $\{S_k\}$ would be Markovian or if $\{X_k\}$ were Markovian, the joint process $\{(X_k, S_k)\}$ would be Markovian. For such Markov sources, there is an almost complete theory of the verification of stochastic stability through the analysis of finite-mean recurrence times to suitably defined sets (atoms, or artificial atoms constructed through *small* or *petite* sets and the splitting technique due to [1, 23]) as well as the regularity properties of the kernel (such as utilizing continuity of the transition kernel and majorization by a finite measure; see, e.g., [21, 15]). For systems of the form (1.1) with only stationary $\{X_k\}$, however, there does not exist a complete theory even though the notion of *renovating events* [3, 4] that is related to the concepts of recurrence and coupling in Markov chains have been utilized in many applications especially in queuing theory.

Such problems arise in many applications in feedback quantization and source coding, networks, and stochastic control. As an example, consider the following scheme which includes the Δ -modulation [7] algorithm commonly used in source coding as a special case: Let $\{X_k\}$ be stationary and ergodic, $Q : \mathbb{R} \rightarrow \mathbb{M} \subset \mathbb{R}, |\mathbb{M}| < \infty$ be a quantizer, and consider the following update:

$$(1.6) \quad S_{k+1} = S_k + Q(X_k - S_k),$$

where $S_0 = 0$. Here, S_k is the output of an adaptive encoder and X_k is the source to be encoded.

In addition to further adaptive coding schemes, applications include stabilization of controlled systems driven by noise processes with memory, design of networked control systems over channels with memory, as well as network and queuing systems.

Such stability problems have been investigated for a number of setups; for an incomplete list see [7, 8, 17, 18, 19, 22, 31, 33]. Notably, the contributions in Kieffer [17, 18] and Kieffer and Dunham [19] are the most relevant ones to the discussion in this paper. These have studied problems motivated from applications in source coding and quantization as in (1.6). [19] considered a non-Markovian setup where \mathbb{S} is countable, [17] considered a setup where \mathbb{S} is not countable, but $f(x, \cdot)$ is continuous on $\mathbb{S}^{\mathbb{N}}$ for every x . Our approach and the proof technique is different than that considered in the literature; notably from that of Kieffer [17], and Kieffer and Dunham [19] (as well as other contributions such as [7, 8, 31] and [22] which can be approached by finite dimensional Markov chain formulations).

Our approach builds on Markov process theoretic techniques where we model the stochastic process $(X_{(-\infty, k]}, S_k)$ or $(X_{[k, \infty)}, S_{[k, \infty)})$ as an infinite dimensional Markov chain. The approach of viewing $(X_{(-\infty, k]}, S_k)$ as a Markov chain, to our knowledge, has been first studied by Hairer [11], where the focus of the author has been on the uniqueness of an invariant measure on the state process S_k , under the assumption that an invariant measure exists and with further regularity assumptions. In this paper, we provide sufficient conditions for the existence of an invariant probability measure

for the joint process while deriving our results. We also establish connections with asymptotic mean stationarity, in addition to the existence of an invariant measure, and ergodicity.

We will see that conditions of the form

$$(1.7) \quad \lim_{M \rightarrow \infty} \left(\limsup_{T \rightarrow \infty} \frac{1}{T} \sum_{k=0}^{T-1} P(|S_k| \geq M) \right) = 0,$$

play an important role for the stochastic stability results in this paper. Even though in the applications we consider we will explicitly study sufficient conditions for such a result, for a class of non-Markovian sources useful sufficient conditions (inspired from applications in queuing and networks) are given in [13] and [24]. These follow from Lyapunov-drift-type conditions such as: $E[(S_{n+1} - S_n)1_{S_n > L} | S_0, \dots, S_n] < -A_1$ with $S_n \geq 0$ for all n for some $A_1 > 0, L < \infty$, and a uniform bound on jumps from below L as $E[|S_{n+1} - S_n|^p | S_0, \dots, S_n] < A_2$ for $p > 1, A_2 > 0$ leading to finite bounds on $\sup_n E[|S_n|^{p-1-\delta}]$ for arbitrarily small $\delta > 0$, which through an application of Markov's inequality lead to (1.7). Thus, the findings in [13] and [24] together with the results in this paper can be used to obtain *Foster-Lyapunov*-type drift criteria for various forms of stochastic stability.

A further related view for approaching such problems is the traditional random dynamical systems view in which one studies the properties of the shifted sequences $(S_{[k,\infty)}, X_{[k,\infty)})$: Such a viewpoint leads to the interpretation that the entire uncertainty is realized in the initial state of the Markov chain, and the process evolves deterministically through a shift map. This approach has led to important contributions on ergodic theory and the introduction of useful notions such as asymptotic mean stationarity [9]. Connections between the two approaches and the implications on the convergences of (1.2)–(1.5) will be made in the paper.

In section 2, we discuss the conditions for the existence of an invariant probability measure. In section 3, we discuss the conditions for asymptotic mean stationarity and ergodicity. This is followed by a study of applications in feedback quantization and networked stochastic control in section 4. Section A in the appendix contains a brief review of Markov chains and ergodic theorems for Markov chains.

2. Stochastic stability of non-Markovian systems. Towards obtaining a method to study such systems, we will here view the process $(X_{(-\infty,k]}, S_k)$ as an $\mathbb{X}^{\mathbb{Z}^-} \times \mathbb{S}$ -valued Markov process, similar to [11]. We recall that with \mathbb{X} a complete, separable, metric (that is, a Polish) space, $\Sigma = \mathbb{X}^{\mathbb{Z}^-}$ is also a Polish space under the product topology.

By a standard argument (e.g., Chapter 7 in [6]), we can embed the one-sided stationary process $\{X_k, k \in \mathbb{Z}_+\}$ into a bilateral (double-sided) stationary process $\{X_k, k \in \mathbb{Z}\}$. We first state the following.

LEMMA 2.1. *The sequence (Z_k, S_k) with $Z_k = X_{(-\infty,k]}$ is a Markov process.*

Proof. For any Borel $A \times B \in \mathcal{B}(\mathbb{X}^{\mathbb{Z}^-} \times \mathbb{S})$ and $k \geq 0$, the following holds almost surely:

$$\begin{aligned} &P((Z_{k+1}, S_{k+1}) \in (A \times B) | Z_m, S_m, m \leq k) \\ &= P((Z_{k+1}, F(X_k, S_k)) \in (A \times B) | Z_m, S_m, m \leq k) \\ &= P((Z_{k+1} \in A) \cap (F(X_k, S_k) \in B) | Z_m, S_m, m \leq k) \\ &= P(X_{(-\infty,k+1]} \in A | Z_m, S_m, m \leq k) 1_{\{F(X_k, S_k) \in B\}} \end{aligned}$$

$$\begin{aligned}
&= P(X_{(-\infty, k+1]} \in A | X_{(-\infty, m]}, S_m, m \leq k) \mathbf{1}_{\{F(X_k, S_k) \in B\}} \\
&= P(X_{(-\infty, k+1]} \in A | X_{(-\infty, k]}) \mathbf{1}_{\{F(X_k, S_k) \in B\}} \\
&= P((X_{(-\infty, k+1]} \in A) \cap (F(X_k, S_k) \in B) | X_{(-\infty, k]}, S_k) \\
&= P((Z_{k+1}, F(X_k, S_k)) \in (A \times B) | Z_k, S_k) \\
&= P((Z_{k+1}, S_{k+1}) \in (A \times B) | Z_k, S_k). \quad \square
\end{aligned}$$

We let \mathbb{P} denote the transition kernel for this process. Our inspiration for taking the approach below builds on the fact that, since X_k is known to be stationary, if there were an invariant measure ν for this process, then this would decompose as

$$\nu(ds_0 | x_{(-\infty, 0]}) \pi(dx_{(-\infty, 0]})$$

with π being the stationary measure for X_k . This follows by the invariance condition:

$$\begin{aligned}
&\int_{\mathbb{X}^{\mathbb{Z}} \times \mathbb{S}} \mathbb{P}(X_{(-\infty, k+1]}, S_{k+1} \in B \times \mathbb{S} | x_{(-\infty, k]}, s_k) \nu(dx_{(-\infty, k]}, ds_k) \\
&= \int_{\mathbb{X}^{\mathbb{Z}} \times \mathbb{S}} P(S_{k+1} \in \mathbb{S} | X_{(-\infty, k+1]} \in B, x_{(-\infty, k]}, s_k) \\
&\quad \times P(X_{(-\infty, k+1]} \in B | x_{(-\infty, k]}, s_k) \nu(dx_{(-\infty, k]}, ds_k) \\
&= \int_{\mathbb{X}^{\mathbb{Z}} \times \mathbb{S}} P(X_{(-\infty, k+1]} \in B | x_{(-\infty, k]}, s_k) \nu(dx_{(-\infty, k]}, ds_k) \\
(2.1) \quad &= \int_{\mathbb{X}^{\mathbb{Z}} \times \mathbb{S}} P(X_{(-\infty, k+1]} \in B | x_{(-\infty, k]}) \nu(dx_{(-\infty, k]}, ds_k) \\
&= \int_{\mathbb{X}^{\mathbb{Z}} \times \mathbb{S}} P(X_{(-\infty, k+1]} \in B | x_{(-\infty, k]}) \nu(dx_{(-\infty, k]}) \\
(2.2) \quad &= \pi(B).
\end{aligned}$$

Here, in (2.1) we use the fact that given $x_{(-\infty, k]}$, to predict X_{k+1} , S_k is noninformative.

2.1. Implications of the existence of an invariant probability measure.

If there is an invariant probability measure \bar{P} for such a process we say that the process is *stochastically stable*. By the ergodic theorem (see Theorem A.1), \bar{P} a.s.

$$(2.3) \quad \lim_{N \rightarrow \infty} \frac{1}{N} E_{x_{(-\infty, 0]}, s} \left[\sum_{k=0}^{N-1} f(X_{(-\infty, k]}, S_k) \right] = f^*(x_{(-\infty, 0]}, s)$$

exists for all measurable and bounded f and for corresponding functions f^* (where the full set of convergence may depend on the function f).

The following assumption will be useful in establishing further stability results in section 3. Recall that $S_0 \sim \kappa$ for some probability measure κ .

Assumption 2.2. The invariant measure \bar{P} is such that $\pi \times \kappa \ll \bar{P}$. That is, $\bar{P}(A, B) = 0$ implies that $\pi(A)\kappa(B) = 0$ for any Borel A, B .

Under this assumption, we would have that the set of initial conditions which may not satisfy (2.3) (this set has zero measure under \bar{P}) also has zero measure under the initial product probability measure $\pi \times \kappa$. Thus,

$$(2.4) \quad \int_{\mathbb{X}^{\mathbb{Z}^-}} \pi(dx) \int_{\mathbb{S}} \kappa(ds_0) E_{x_{(-\infty,0]}, s_0} \left[\frac{1}{T} \left[\sum_{k=0}^{T-1} g(x_{(-\infty,k]}, s_k) \right] \right] \\ \rightarrow \int_{\mathbb{X}^{\mathbb{Z}^-}} \int_{\mathbb{S}} \kappa(ds_0) \pi(dx) f^*(x_{(-\infty,0]}, s_0).$$

Furthermore, by Theorem A.2, sample paths also converge almost surely. Thus, convergence in the sense of (1.2), (1.3), and (1.5) will hold. We will later see that (1.4) will also hold.

2.2. Existence of an invariant probability measure with finite \mathbb{S} . Our first result is for the setup with finite \mathbb{S} . For some related results and an alternative approach for the finite case, see [20].

THEOREM 2.3. *Consider the dynamical system given by (1.1). Suppose that \mathbb{S} is finite. Then, the process is stochastically stable.*

Proof. Define for all $a \in \mathbb{S}$, the sequence of expected occupational measures

$$(2.5) \quad v_t(dx_{(-\infty,k]} \times \{a\}) = E \left[\frac{1}{T} \sum_{k=0}^{t-1} 1_{\{X_{(-\infty,k]}, S \in dx_{(-\infty,k]} \times \{a\}\}} \right] \\ = \frac{1}{T} \sum_{k=0}^{t-1} P(X_{(-\infty,k]}, S \in dx_{(-\infty,k]} \times \{a\}),$$

where for every k , $X_{(-\infty,k]} \sim \pi$. Thus, for any t , we can decompose $v_t(dx_{(-\infty,k]} \times \{a\}) = \pi(dx_{(-\infty,k]})v_t(a|x_{(-\infty,k]})$ since π is a stationary measure by (2.2). It follows then that

$$v_t(dx_{(-\infty,k]} \times \{a\}) \leq \pi(dx_{(-\infty,k]})$$

for every a and $|\mathbb{S}|\pi(dx_{(-\infty,k]})$ is a majorizing finite measure for the sequence v_t .

By [15, Proposition 1.4.4], a sequence of probability measures which is uniformly countable additive is setwise sequentially precompact (see pp. 6–8 in [15]), a sufficient condition being that the sequence is majorized by a finite measure. Thus, $\{v_t\}$ has a converging subsequence v_{t_k} so that for some probability measure v , $v_{t_k}(A) \rightarrow v(A)$ for all Borel A . Let \mathbb{P} be the transition kernel for the Markov chain. Then through a Krylov–Bogoliubov-type argument, for every Borel A ,

$$(2.6) \quad |v_N(A) - v_N \mathbb{P}(A)| \\ = \left| \frac{1}{N} \left((v_0(A) + \dots + v \mathbb{P}^{(N-1)}(A)) - (v_0 \mathbb{P}(A) + \dots + v \mathbb{P}^N(A)) \right) \right| \\ \leq \frac{1}{N} |v_0(A) - v_0 \mathbb{P}^N(A)| \rightarrow 0.$$

Since $v_{t_k} \rightarrow v$, setwise, it follows that $v_{t_k} \mathbb{P}(B) \rightarrow v \mathbb{P}(B)$ also and hence $v(B) = v \mathbb{P}(B)$ and v is stationary.

Hence, there exists an invariant probability measure for the process. □

2.3. Existence of an invariant probability measure with countable \mathbb{S} . In this section, we assume that \mathbb{S} is a countable set viewed as a subset of \mathbb{R} whose elements are uniformly separated from each other; thus \mathbb{S} is a uniformly discrete set in the sense that there exists $r > 0$ such that $|x - y| > r$ for all $x, y \in \mathbb{S}$.

THEOREM 2.4. *Consider the dynamical system given by (1.1). If (1.7) holds with the norm defined on \mathbb{R} , the process is stochastically stable.*

Proof.

- (i) The sequence v_t defined in (2.5) is tight: As an individual probability measure, π is tight. Since by (1.7), the sequence of marginals of $v_t^{\mathbb{S}}$ on \mathbb{S} is also tight, it follows that the product measure is also tight: For every $\epsilon > 0$, there exists a compact set $\mathcal{L} \times \mathcal{M}$ in the product space so that

$$v_t((\mathcal{L} \times \mathcal{M})^C) \leq \pi(\mathcal{L}^C) + v_t^{\mathbb{S}}(\mathcal{M}^C) \leq \epsilon.$$

This follows since $(\mathcal{L} \times \mathcal{M})^C = (\mathcal{L}^C \times \mathbb{S}) \cup (\mathbb{X}^{\mathbb{Z}^-} \times \mathcal{M}^C)$, where for a set A , A^C denotes its complement.

- (ii) We show that the sequence v_t is relatively compact under the w -s topology [2, 27]: Let \mathbb{A}, \mathbb{B} be complete, separable, metric spaces. The w -s topology on the set of probability measures $\mathcal{P}(\mathbb{A} \times \mathbb{B})$ is the coarsest topology under which $\int f(a, b)\nu(da, db) : \mathcal{P}(\mathbb{A} \times \mathbb{B}) \rightarrow \mathbb{R}$ is continuous for every measurable and bounded f which is continuous in $b \in \mathbb{B}$ for every $a \in \mathbb{A}$ (but unlike weak topology, f does not need to be continuous in a).

Since the marginals on $x_{(-\infty, k]}$ are fixed, [27, Theorem 3.10] (see also [2, Theorem 2.5]) establishes that the set of such measures is relatively compact under the w -s topology when a tightness condition holds. By tightness from (i), let v_{t_k} be a w -s converging subsequence of v_t . Then, as in (2.6), for every Borel A ,

$$\begin{aligned} (2.7) \quad & |v_N(A) - v_N\mathbb{P}(A)| \\ &= \frac{1}{N} |(v_0(A) + \dots + v\mathbb{P}^{(N-1)}(A)) - (v_0\mathbb{P}(A) + \dots + v\mathbb{P}^N(A))| \\ &\leq \frac{1}{N} |v_0(A) + v_0\mathbb{P}^N(A)| \rightarrow 0. \end{aligned}$$

Now, v_{t_k} converges w -s to v for some v . In particular, for every measurable and bounded function f (which is continuous in s since \mathbb{S} is countable), it holds that $\langle v_{t_k}, f \rangle \rightarrow \langle v, f \rangle$, where $\langle v_{t_k}, f \rangle := \int v_{t_k}(dx, s)f(x, s)$. We wish to show that $\langle v_{t_k}\mathbb{P}, f \rangle \rightarrow \langle v\mathbb{P}, f \rangle$, leading to the invariance of v in view of (2.7). Now, let f be measurable and bounded. We have that

$$\int v_{t_k}(dx, s)f(x, s) \rightarrow \int v(dx, s)f(x, s).$$

Observe that the transitioned probability measure $v_{t_k}\mathbb{P}$ satisfies, for every such f ,

$$\begin{aligned} (2.8) \quad & \int v_{t_k}\mathbb{P}(dx, s)f(x, s) \\ &= \int \pi(dz) \left(\sum_{s'} v_{t_k}(s'|z) \int_x P(dx|z)f(x, F(z, s')) \right), \end{aligned}$$

where $P(dx|z) = P(X_{(-\infty, k+1]} \in dx | X_{(-\infty, k]} = z)$. With $z = x_{(-\infty, k]}$, let

$$\mathbb{P}f(z, s) = \int_x P(dx|z)f(x, F(x_k, s')) =: g(z, s).$$

We note here that with z specified, x_k is determined. The measurable function g is continuous in s for every x . This ensures that $\langle v_{t_k}\mathbb{P}, f \rangle = \langle v_{t_k}\mathbb{P}, f \rangle \rightarrow \langle v\mathbb{P}, f \rangle$ and, for all bounded f continuous in s , $\langle v, f \rangle = \langle v\mathbb{P}, f \rangle$, and hence v is invariant. \square

2.4. Existence of an invariant probability measure with $\mathbb{S} = \mathbb{R}^n$. We have the following assumption.

Assumption 2.5. $F(x, s)$ is continuous in s for every x .

THEOREM 2.6. *Consider the dynamical system given by (1.1). If (1.7) holds, under Assumption 2.5, the process is stochastically stable.*

Proof.

- (i) As in the proof of Theorem 2.4(i), the sequence defined in (2.5) with the set $\{a\}$ replaced with an arbitrary Borel set B is tight.
- (ii) As before in Theorem 2.4(ii), the sequence v_t is relatively compact under the w - s topology. Since the marginals on $x_{(-\infty, k]}$ are fixed, [27, Theorem 3.10] establishes that the set of strategic measures is relatively compact under the w - s topology under tightness. By tightness, let v_{t_k} be a w - s converging subsequence of v_t . Then, as in (2.6), for every Borel A ,

$$\begin{aligned}
 (2.9) \quad & |v_N(A) - v_N \mathbb{P}(A)| \\
 &= \left| \frac{1}{N} \left(v_0(A) + \dots + v \mathbb{P}^{(N-1)}(A) \right) - (v_0 \mathbb{P}(A) + \dots + v \mathbb{P}^N(A)) \right| \\
 &\leq \frac{1}{N} (v_0(A) + v \mathbb{P}^N(A)) \rightarrow 0.
 \end{aligned}$$

Now, v_{t_k} converges w - s to v for some v . In particular, for every measurable and bounded function f which is furthermore continuous in s for every x , it holds that $\langle v_{t_k}, f \rangle \rightarrow \langle v, f \rangle$, where $\langle v_{t_k}, f \rangle := \int v_{t_k}(dx, ds) f(x, s)$. We will show that $\langle v_{t_k} \mathbb{P}, f \rangle \rightarrow \langle v \mathbb{P}, f \rangle$, leading to the invariance of v in view of (2.9). Now, let f be measurable and bounded, but continuous in s . We have that

$$\int v_{t_k}(dx, ds) f(x, s) \rightarrow \int v(dx, ds) f(x, s).$$

Observe that the (transitioned) probability measure $v_{t_k} \mathbb{P}$ satisfies, for every measurable bounded f continuous in s for every x ,

$$\begin{aligned}
 (2.10) \quad & \int v_{t_k} \mathbb{P}(dx, ds) f(x, s) \\
 &= \int \pi(dz) \left(\int_{s'} v_{t_k}(ds'|z) \int_x P(dx|z) f(x, F(s', z)) \right),
 \end{aligned}$$

where $P(dx|z) = P(X_{(-\infty, k+1]} \in dx | X_{(-\infty, k]} = z)$. With $z = x_{(-\infty, k]}$, let

$$\mathbb{P}f(z, s) = \int_x P(dx|z) f(x, F(x_k, s')) =: g(z, s).$$

We note here that with z specified, x_k is determined. Since F is continuous in s' and f is continuous in s , by the dominated convergence theorem g is continuous in s for every z . Thus, \mathbb{P} preserves w - s continuity and thus $\langle v_{t_k}, \mathbb{P}f \rangle = \langle v_{t_k} \mathbb{P}, f \rangle \rightarrow \langle v \mathbb{P}, f \rangle$ and for all bounded f continuous in s : $\langle v, f \rangle = \langle v \mathbb{P}, f \rangle$, and hence v is invariant. □

2.5. Existence of an invariant probability measure under quasi-continuity conditions. A large class of applications do not have the property that \mathbb{S} is countable or that F is continuous in s . To approach such problems in our framework, we impose the following quasi-Feller-type condition which is natural for the applications we will consider.

Assumption 2.7. $F(x, s)$ is continuous on $\mathbb{X} \times \mathbb{S} \setminus D$, where D is a closed set with $P((X_{t+1}, S_{t+1}) \in D | x_{(-\infty, t]} = x, s_t = s) = 0$ for all x, s . Furthermore, with $D_\epsilon = \{z : d(z, D) < \epsilon\}$ for $\epsilon > 0$ and d the product metric on $\mathbb{X} \times \mathbb{S}$, for some $K < \infty$, we have that for all x, s , and $\epsilon > 0$

$$P((X_{t+1}, S_{t+1}) \in D_\epsilon | x_{(-\infty, t]} = x, s_t = s) \leq K\epsilon.$$

Note that we can write the above as

$$P(((x_{(-\infty, t]}, X_{t+1}), S_{t+1}) \in \{x_{(-\infty, t]}\} \times D_\epsilon | x_{(-\infty, t]} = x, s_t = s) \leq K\epsilon.$$

Furthermore, this is equivalent to the condition

$$P(((x_{(-\infty, t]}, X_{t+1}), S_{t+1}) \in \mathbb{X}^{\mathbb{Z}^-} \times D_\epsilon | x_{(-\infty, t]} = x, s_t = s) \leq K\epsilon.$$

We define $\mathbb{D}_\epsilon := \mathbb{X}^{\mathbb{Z}^-} \times D_\epsilon$. This is an open set in $\mathbb{X}^{\mathbb{Z}^-} \times \mathbb{S}$ and will be useful in the analysis to follow.

We remark that Assumption 2.7 is related to what is referred to as the *quasi-Feller* condition introduced by Lasserre (see [15, section 7.3]). Our definition here and the proof is different in part because we do not assume that the state space is locally compact. We have the following assumption.

Assumption 2.8.

- (i) If \mathbb{X} is compact, $\int_{\mathbb{X}} P(X_{k+1} \in dx | X_{(-\infty, k]} = z) f(x)$ is continuous in z for every continuous and bounded f on \mathbb{X} .
- (ii) If \mathbb{X} is not compact, $\int_{\mathbb{X}} P(X_{k+1} \in dx | X_{(-\infty, k]} = z) f(x)$ is continuous in z for every measurable and bounded f on \mathbb{X} .

THEOREM 2.9. *Suppose that Assumptions 2.7 and 2.8 hold. If (1.7) holds, the system is stochastically stable.*

Proof. Assumption 2.7 implies that every converging subsequence v_{n_k} of

$$v_n(A \times B) = E_{x,s} \left[\frac{1}{N} \sum_{k=0}^{n-1} 1_{\{x_{(-\infty, k]}, s_k \in (A \times B)\}} \right]$$

is such that for all $\epsilon > 0$

$$\limsup_{n_k \rightarrow \infty} v_{n_k}(\mathbb{D}_\epsilon) \leq K\epsilon.$$

Note that with $v = \lim_{n_k \rightarrow \infty} v_{n_k}$, it follows from the Portmanteau theorem (see, e.g., [5, Thm.11.1.1]) that

$$v(\mathbb{D}_\epsilon) \leq K\epsilon.$$

Now, consider a weakly converging empirical occupation sequence v_{t_k} and let this sequence have an accumulation point v^* . We will show that v^* is invariant.

Observe that the transitioned probability measure $v_{t_k}\mathbb{P}$ satisfies, for every continuous and bounded f ,

$$(2.11) \quad \int v_{t_k}\mathbb{P}(dx, ds)f(x, s) = \int \pi(dz) \left(\int_{s'} v_{t_k}(ds'|z) \int_x P(dx|z)f(x, F(s', z)) \right),$$

where $P(dx|z) = P(X_{(-\infty, k+1]} \in dx | X_{(-\infty, k]} = z)$. With $z = x_{(-\infty, k]}$, let

$$\mathbb{P}f(z, s) = \int_x P(dx|z)f(x, F(x_k, s)) =: g(z, s).$$

We note here that with z specified, x_k is determined. In the following, we show that for continuous and bounded f , g is continuous whenever F is (thus outside \mathbb{D}_ϵ).

If \mathbb{X} is compact, by Tychonoff's theorem $\mathbb{X}^{\mathbb{Z}^-}$ is locally compact. In this case, we will invoke [28, Theorem 3.5] for the following argument. If $z^n \rightarrow z$,

$$f((z^n, x_{k+1}), F(x_k^n, s)) \rightarrow f((z, x_{k+1}), F(x_k, s))$$

for every x_{k+1} and thus with $H_{s,n}(x_{k+1}) := f((z^n, x_{k+1}), F(x_k^n, s))$, $H_s(x_{k+1}) := f((z, x_{k+1}), F(x_k, s))$, it follows that $H_{s,n}(x_{k+1}) \rightarrow H_s(x_{k+1})$ as $z_{k+1}^n \rightarrow z_{k+1}$; thus, we have *continuous convergence* as it is defined in [28]. As a result, continuity of g (outside \mathbb{D}_ϵ) is established by a generalized dominated convergence theorem given in [28, Theorem 3.5] in view of weak continuity by Assumption 2.8(i).

If \mathbb{X} is not compact, we invoke the generalized dominated convergence theorem of [28, Theorem 2.4]: Since P is strongly continuous under Assumption 2.8(ii), and f is continuous and bounded, $g(z, s) = \int_x P(dx|z)f(x, F(x_k, s'))$ is continuous outside \mathbb{D}_ϵ .

Now, consider $\langle v_{t_k}, \mathbb{P}f \rangle = \langle v_{t_k}, g_f \rangle + \langle v_{t_k}, \mathbb{P}f - g_f \rangle$, where g_f is a continuous function which is equal to $\mathbb{P}f$ outside an open neighborhood of D and is continuous with $\|g_f\|_\infty = \|\mathbb{P}f\|_\infty \leq \|f\|_\infty$. The existence of such a function follows from the Tietze-Urysohn extension theorem [5], where the closed set is given by $\mathbb{X}^{\mathbb{Z}^-} \times \mathbb{S} \setminus \mathbb{D}_\epsilon$. It then follows from Assumption 2.7 that, for every $\epsilon > 0$, a corresponding g_f can be found so that $\langle v_{t_k}, \mathbb{P}f - g_f \rangle \leq K\|f\|_\infty\epsilon$, and since $\langle v_{t_k}, g_f \rangle \rightarrow \langle v^*, g_f \rangle$, it follows that

$$(2.12) \quad \begin{aligned} \limsup_{t_k \rightarrow \infty} |\langle v_{t_k}, \mathbb{P}f \rangle - \langle v^*, \mathbb{P}f \rangle| &= \limsup_{t_k \rightarrow \infty} |\langle v_{t_k}, \mathbb{P}f - g_f \rangle - \langle v^*, \mathbb{P}f - g_f \rangle| \\ &\leq \limsup_{t_k \rightarrow \infty} |\langle v_{t_k}, \mathbb{P}f - g_f \rangle| + |\langle v^*, \mathbb{P}f - g_f \rangle| \\ &\leq 2K'\epsilon. \end{aligned}$$

Here, $K' = 2K\|f\|_\infty$ is fixed and ϵ may be made arbitrarily small. We conclude that v^* is invariant. □

Remark 2.10. In his definition for quasi-Feller chains, Lasserre assumes the state space to be locally compact. A product space is locally compact if the individual coordinate spaces are compact by Tychonoff's theorem and the state space in our formulation is not locally compact, in general. However, we invoke tightness directly with no use of convergence properties of the set of functions which decay to zero as is done in [15]. We also note that Gersho [7] had obtained a similar result addressing points of discontinuity in the context of adaptive quantizer design.

3. Asymptotic mean stationarity and ergodicity.

3.1. Shifts and random dynamical systems view. As an alternative approach, we may also view $X_{[k,\infty)}, S_{[k,\infty)}$ as an infinite dimensional Markov chain. This viewpoint is more commonly adopted in the information theory literature (even though not explicitly stated as a Markov chain), as we discuss in the following. Such a view gives rise to important notions such as asymptotic mean stationarity. Note that such a viewpoint leads to the interpretation that the entire uncertainty is realized in the initial state of the Markov chain, and the process evolves deterministically.

Let \mathbb{X} be a complete, separable, metric space. Let $\mathcal{B}(\mathbb{X})$ denote the Borel sigma-field of subsets of \mathbb{X} , let $\Sigma = \mathbb{X}^{\mathbb{Z}^+}$ denote the sequence space of all one-sided (unilateral) infinite sequences drawn from \mathbb{X} . Thus, if $x \in \Sigma$ then $x = \{x_0, x_1, x_2, \dots\}$ with $x_i \in \mathbb{X}$. Let $X_n : \Sigma \rightarrow \mathbb{X}$ denote the coordinate function such that $X_n(x) = x_n$. Let T denote the shift operation on Σ , that is, $X_n(Tx) = x_{n+1}$.

With \mathbb{X} a Polish space, $\Sigma = \mathbb{X}^{\mathbb{Z}^+}$ is also a Polish space under the product topology. Let $\mathcal{B}(\Sigma)$ denote the smallest σ -field containing all cylinder sets which are of the form $\{x : x_i \in B_i, m \leq i \leq n\}$, where $B_i \in \mathcal{B}(\mathbb{X})$, for all integers $m, n \geq 0$. Here, $\bigcap_{n \geq 0} T^{-n} \mathcal{B}(\Sigma)$ is the tail σ -field: $\bigcap_{n \geq 0} \sigma(X_n, X_{n+1}, \dots)$, since $T^{-n}(A) = \{x : T^n x \in A\}$.

Let μ be the measure on the process $\{X_0, X_1, \dots\}$. This process is stationary and μ is said to be a stationary (or invariant) measure on $(\Sigma, \mathcal{B}(\Sigma))$ if $\mu(T^{-1}B) = \mu(B)$ for all $B \in \mathcal{B}(\Sigma)$. This random process is ergodic if $A = T^{-1}A$ implies that $\mu(A) \in \{0, 1\}$.

DEFINITION 3.1 (see [10]). *A process on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with process measure μ , is asymptotically mean stationary (AMS) if there exists a probability measure \bar{P} such that*

$$(3.1) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} \mu(T^{-k}F) = \bar{P}(F)$$

for all events $F \in \mathcal{B}(\Sigma)$. Here \bar{P} is called the stationary mean of μ , and is a stationary measure.

\bar{P} is stationary since, by definition $\bar{P}(F) = \bar{P}(T^{-1}F)$.

As elaborated on earlier, we may view $\{T^n X\}$ as a Markov chain (whose only uncertainty is hidden in the initial distribution) characterized by a transition function in the following, taking values in the Polish space $\mathbb{X}^{\mathbb{Z}^+}$. The kernel is given such that, for every $n \in \mathbb{N}$,

$$\int \mu(dx) P^n(x, B) = \mu(T^n X \in B) = \mu(X \in T^{-n}B).$$

We may define expected empirical occupation measures as follows:

$$(3.2) \quad v_t(B) = \frac{1}{T} E_\mu \left[\sum_{k=0}^{t-1} 1_{\{T^k X \in B\}} \right] := \frac{1}{T} \sum_{k=0}^{t-1} \mu(X \in T^{-k}B).$$

In this case, the initial measure μ on the sequence space affects how convergence occurs. If the time averages converge *setwise* as in (3.1) to some measure \bar{P} , the process is AMS.

It follows that if the system is AMS, (1.4) holds since the set of simple functions is dense in the set of measurable and bounded functions under the supremum norm.

3.2. Sufficient conditions for asymptotic mean stationarity. It is an important question to ask when a process is AMS.

THEOREM 3.2 (see [26]). *A process is AMS if and only if it is asymptotically dominated by a stationary process, that is there exists a stationary measure π such that for Borel B if $\pi(B) = 0$ then $\lim_{n \rightarrow \infty} \mu(T^{-n}B) = 0$.*

Due to the Markov formulation, we can obtain the following direct condition to check whether the AMS property holds for systems of the form (1.1).

THEOREM 3.3. *Let there exist a stationary measure \bar{P} for the Markov chain $(X_{(-\infty,k]}, S_k)$ for the system (1.1). Assumption 2.2 implies the AMS property for the process (X_k, S_k) .*

Proof. We will arrive at the conclusion using an ergodic theoretic result. From the last item of Theorem A.3, if $\pi \times \kappa \ll \bar{P}$, the following uniform convergence holds:

$$(3.3) \quad \lim_{T \rightarrow \infty} \sup_{f \in M(\mathbb{X}^{\mathbb{Z}^-} \times \mathbb{S}) : \|f\|_\infty \leq 1} \left| E_{\pi \times \kappa} \frac{1}{T} \left[\sum_{k=0}^{T-1} f(X_{(-\infty,k]}, S_k) \right] - \eta^*(f) \right| = 0$$

for some invariant measure η^* (not necessarily equal to \bar{P}), where $M(\mathbb{X}^{\mathbb{Z}^-} \times \mathbb{S})$ denotes the set of measurable and bounded functions on $\mathbb{X}^{\mathbb{Z}^-} \times \mathbb{S}$.

We will now see that the above implies the AMS property. We will obtain the result for a general Markov process taking values in some Polish space \mathbb{V} rather than the $\mathbb{X}^{\mathbb{Z}^-} \times \mathbb{S}$ -valued process considered, for ease in presentation. Let the initial measure be ν and the resulting measure on the state space $\mathbb{V}^{\mathbb{Z}^+}$ be \mathbb{P}_ν . Let ν^* be an invariant probability measure for the Markov chain and \mathbb{P}_{ν^*} be the resulting stationary measure on the product space $\mathbb{V}^{\mathbb{Z}^+}$. Consider a set $A \in \mathcal{B}(\mathbb{V}^{\mathbb{Z}^+})$ and let A_1 be a corresponding open finite dimensional set so that $x \in A$ is equivalent to $x_{[0,m]} \in A_1$ for some m . Then,

$$\begin{aligned} E_{\mathbb{P}_\nu} \frac{1}{T} \left[\sum_{k=0}^{T-1} 1_{\{v_{k,\infty}\} \in A\}} \right] &= E_{\mathbb{P}_\nu} \frac{1}{T} \left[\sum_{k=0}^{T-1} 1_{\{v_{k,k+m}\} \in A_1\}} \right] \\ &= E_{\mathbb{P}_\nu} \left[\frac{1}{T} \sum_{k=0}^{T-1} E[1_{\{v_{k,k+m}\} \in A_1\}} | v_{[0,k]}] \right] \\ &= E_{\mathbb{P}_\nu} \frac{1}{T} \left[\sum_{k=0}^{T-1} E \left[1_{\{v_{k,k+m}\} \in A_1\}} | v_k \right] \right] \\ &= E_{\mathbb{P}_\nu} \frac{1}{T} \left[\sum_{k=0}^{T-1} g(v_k) \right] = E_{\nu_0=\nu} \frac{1}{T} \left[\sum_{k=0}^{T-1} g(v_k) \right] \\ &\rightarrow \int \eta^*(dv) g^*(v) = \int \eta^*(dv) g(v) \\ (3.4) \quad &= E_{\mathbb{P}_{\eta^*}} \frac{1}{T} \left[\sum_{k=0}^{T-1} 1_{\{v_{k,\infty}\} \in A\}} \right] = \mathbb{P}_{\eta^*}(A) \end{aligned}$$

for some invariant measure η^* and measurable g^* . Here, $g(v_k) = E[1_{\{v_{k,k+m}\} \in A_1\}} | v_k]$. The first equality above follows from the fact that A is a finite dimensional cylinder set and A_1 is the corresponding finite dimensional set, the second equality from the iterated expectations, the third from the fact that v_k is Markov. Since, $g(v)$ is measurable and bounded, (3.3) leads to the desired result. \square

Naturally, if \bar{P} is the unique invariant measure, $\eta^* = \bar{P}$ in (3.3). Next, we study the uniqueness problem.

3.3. Ergodicity. Ergodicity is a desirable stability property, since it allows for the sample path averages to converge to the same limit in the ergodic theorem regardless of the initial distribution, leading to crucial consequences in information theoretic and control theoretic applications. For a Markov chain, the uniqueness of an invariant probability measure implies ergodicity (see, e.g., [15, Chap. 2]). With the random dynamical systems view, sufficient conditions such as mixing can be utilized [9], however, these may be restrictive.

Consider an \mathbb{X} -valued Markov chain with transition kernel P , where \mathbb{X} is a complete, separable, and metric space.

DEFINITION 3.4. *A Markov chain is μ -irreducible, if for any set $B \in \mathcal{B}(\mathbb{X})$ such that $\mu(B) > 0$, and $\forall x \in \mathbb{X}$, there exists some integer $n > 0$, possibly depending on B and x , such that $P^n(x, B) > 0$, where $P^n(x, B)$ is the transition probability in n stages from x to B .*

A maximal irreducibility measure ψ is an irreducibility measure such that, for all other irreducibility measures ϕ , we have $\psi(B) = 0 \Rightarrow \phi(B) = 0$ for any $B \in \mathcal{B}(\mathcal{X})$ (that is, all other irreducibility measures are absolutely continuous with respect to ψ). Whenever a chain is said to be irreducible, irreducibility with respect to a maximal irreducibility measure is implied. A maximal irreducibility measure ψ exists for a μ -irreducible Markov chain; see [21, Proposition 4.2.4]. The following is a well-known result.

THEOREM 3.5. *Let $\{X_t\}$ be a ψ -irreducible Markov chain which admits an invariant probability measure. The invariant measure is unique.*

A complementary condition for ergodicity is the following.

DEFINITION 3.6. *For a Markov chain with transition kernel P , a point x is accessible if for every y and every open neighborhood O of x , there exists $k > 0$ such that $P^k(y, O) > 0$.*

One can show that if a point is accessible, it belongs to the (topological) support of every invariant measure (see, e.g., Lemma 2.2 in [12]). Recall that the support (or spectrum) of a probability measure is defined to be the set of all points x for which every open neighborhood of x has positive measure.

We recall that a Markov chain V_t is said to have the strong Feller property if $E[f(V_{t+1})|V_t = v]$ is continuous in v for every measurable and bounded f .

THEOREM 3.7 (see [12, 25]). *If a Markov chain over a Polish space has the strong Feller property, and if there exists an accessible point, then the chain can have at most one invariant probability measure.*

However, a Markov chain defined as $(X_{(-\infty, k]}, S_k)$ cannot be strongly Feller due to the memory in the source: Take $f(x) = 1_{\{x_{-1} \in A\}}$ (where $x = x_{(-\infty, 0]}$) for some closed set A , then $E[f(X_{(-\infty, k+1]})|x_{(-\infty, k]}] = 1_{\{x_k \in A\}}$ is not continuous.

Nonetheless, we can have the following slight generalization.

THEOREM 3.8. *Suppose that $E[f(X_{k+1}, S_{k+1})|X_{(-\infty, k]} = x_{(-\infty, k]}, S_k = s_k]$ for measurable and bounded $f : \mathbb{X} \times \mathbb{S} \rightarrow \mathbb{R}$, is continuous in $(x_{(-\infty, k]}, s_k)$. Suppose further that there exists an accessible point for the Markov chain $\{(X_{(-\infty, k]}, S_k)\}$. The chain can have at most one invariant probability measure.*

Proof. As in [12], suppose there exist two invariant probability measures π_1, π_2 both of which must include (x, s) in their topological supports. Then, there exist disjoint sets U and V and probability measures $\tilde{\pi}_1$ and $\tilde{\pi}_2$ so that $\tilde{\pi}_1(U) = 1$ and $\tilde{\pi}_2(V) = 1$ (see, e.g., [15, Lemma 2.2.3]). Now, there cannot exist a Borel $A \subset \mathbb{X} \times \mathbb{S}$ such that under one measure it puts

$$\begin{aligned} P((x_{(-\infty,t]}, X_{t+1}, S_{t+1}) \in (\{x_{(-\infty,t]}\} \times A) | x_{(-\infty,t]}, s) \\ = P((X_{t+1}, S_{t+1}) \in A | x_{(-\infty,t]}, s) = 1\tilde{\pi}_1 a.s. \end{aligned}$$

and $P((X_{t+1}, S_{t+1}) \in A | x_{(-\infty,t]}, s) = 0\tilde{\pi}_2$ a.s. since the function $E[1_{\{(X_{t+1}, S_{t+1}) \in A\}} | x, s]$ is continuous in x, s and if a continuous function is a constant $\tilde{\pi}_i$ almost everywhere, then it should be a constant in the topological support of the probability measure. Now, let $\tilde{\pi}_1$ have support U and let $\tilde{\pi}_2$ have support V with $U \cap V = \emptyset$. Then, there exists a set $C \subset \mathbb{X} \times \mathbb{S}$ so that, $P((X_1, S_1) \in C | x_{(-\infty,0]}, s_0) = 1\tilde{\pi}_1$ a.s. and $0\tilde{\pi}_2$ a.s., since the sets U, V would be disjoint absorbing sets. But this leads to a contradiction in view of continuity. \square

For applications such as Δ -modulation, however, we will see that the continuity assumption in Theorem 3.8 fails to hold. To be able to apply the result for such setups, we have the following relaxation.

THEOREM 3.9. *Suppose that for measurable and bounded $f : \mathbb{X} \times \mathbb{S} \rightarrow \mathbb{R}$, $E[f(X_{k+1}, S_{k+1}) | X_{(-\infty,k]}, x_{(-\infty,k]}, S_k = s_k]$ is continuous in $x_{(-\infty,k]}, s_k$ for all $x, s \in (\mathbb{X}^{\mathbb{Z}^-} \times \mathbb{S}) \setminus D$ for some closed set D . Suppose further that there exists an accessible point $(x, s) \notin D$ for the Markov chain $\{(X_{(-\infty,k]}, S_k)\}$. The chain can have at most one invariant probability measure.*

Proof. The proof follows from that of Theorem 3.8, despite the presence of the discontinuous set D ; since the set of continuity $(\mathbb{X}^{\mathbb{Z}^-} \times \mathbb{S}) \setminus D$ is open and contains the accessible point, the proof follows as before. \square

4. Applications. In this section, we consider applications in feedback quantization and networked control.

4.1. Adaptive Quantization. Adaptive quantization for stationary sources has been studied, in particular in [17, 19, 7]. This paper generalizes the results of [7] which investigated Δ -modulation only for finite order Markov sources. We believe that the approach in this paper is more accessible than the arguments in [17] and [19] in part because it allows for, through a unified approach, a Markov chain theoretic approach and also leads to an ergodicity analysis in addition to asymptotic mean stationary.

4.1.1. Δ -modulation.

THEOREM 4.1. *Let X_k be a stationary and ergodic \mathbb{R} -valued process stationary process measure π , $Q : \mathbb{R} \rightarrow \{-m, m\}$, with the following update:*

$$S_{k+1} = S_k + Q(X_k - S_k),$$

where $S_0 = 0$ and $Q(Z) = m1_{\{Z \geq 0\}} - m1_{\{Z < 0\}}$. Suppose further that $E[Q(X_0 - m)] < 0$ and $E[Q(X_0 + m)] > 0$ (equivalently, $P(X_0 \geq m) < 1/2, P(X_0 \leq -m) < 1/2$). Then, the system is stochastically stable in the sense that there exists an invariant probability measure. Furthermore, if for every m, k , and nonempty open A_k , $\pi(X_{[m,k]} \in \prod_{t=m}^k A_k) > 0$, the system is AMS. If, in addition, $E[g(X_1) | x_{(-\infty,0]}]$ is continuous in $x_{(-\infty,0]}$ for measurable and bounded g , (X_k, S_k) is ergodic.

Note that here $\mathbb{S} = \{km, k \in \mathbb{Z}\}$ is a countable set.

An example where the *Lebesgue-irreducibility*-type condition given by $(\pi(X_{[m,k]} \in \prod_{t=m}^k A_k) > 0)$ holds is

$$X_{t+1} = \sum_{i=0}^{\infty} \alpha_i W_{t-i}$$

with $\sum_t |\alpha_t^2| < \infty$ and W_t is a sequence of i.i.d. Gaussian random variables.

An example where the continuity condition for ergodicity holds is the following autoregressive representation

$$X_{t+1} = \sum_{i=0}^{N-1} \alpha_i X_{t-i} + W_t$$

with the roots of $1 - \sum_{i=1}^N \alpha_{i-1} z^{-i}$ strictly inside the unit circle and W_t a sequence of i.i.d. Gaussian random variables. This follows since

$$E[g(X_1)|x_{(-\infty,0]}] = \int g(z) \eta \left(z - \sum_{i=0}^{N-1} \alpha_i x_{-i} \right) dz,$$

with η denoting the Gaussian density and by an application of the dominated convergence theorem, this expression is continuous in $x_{(-\infty,0]}$.

Proof. Observe first that

$$\begin{aligned} & \lim_{M \rightarrow \infty} \left(\limsup_{T \rightarrow \infty} \frac{1}{T} \sum_{k=0}^{T-1} P(|S_k| \geq M) \right) \\ &= \lim_{M \rightarrow \infty} \left(\limsup_{T \rightarrow \infty} \frac{1}{T} \sum_{k=0}^{T-1} (1 - P(|S_k| < M)) \right) \\ &= \lim_{M \rightarrow \infty} \left(1 - \left(\liminf_{T \rightarrow \infty} \frac{1}{T} \sum_{k=0}^{T-1} P(|S_k| < M) \right) \right) \\ &= 1 - \lim_{M \rightarrow \infty} \left(\liminf_{T \rightarrow \infty} \frac{1}{T} \sum_{k=0}^{T-1} P(|S_k| < M) \right) \end{aligned}$$

and thus (1.7) can be equivalently written as

$$(4.1) \quad \lim_{M \rightarrow \infty} \left(\liminf_{T \rightarrow \infty} \frac{1}{T} \sum_{k=0}^{T-1} P(|S_k| < M) \right) = 1.$$

Using the fact that by the ergodicity of the source the following hold a.s.,

$$\begin{aligned} & \lim_{n \rightarrow \infty} (m|\{-n \leq -i : |X_i| \leq m\}| - m|\{-n \leq -i : |X_i| > m\}|) = \infty, \\ & \lim_{n \rightarrow \infty} (-m|\{-n \leq -i : |X_i| \leq m\}| + m|\{-n \leq -i : |X_i| > m\}|) = -\infty, \end{aligned}$$

Kieffer and Dunham [19, Theorem 2] show that the coding scheme satisfies the condition [19, eq. (2.2)], which in turn implies (4.1). This follows since, with K a finite set, the condition $S_i \in K$ for some $i \in \{n, \dots, n + N\}$ [19, eq. (2.2)] implies that $|S_i| \leq K_1 + Nm \leq K_2N$ for constants K_1, K_2 since $|S_i - S_j| \leq |i - j|m$. By Theorem 2.6, the system is stochastically stable.

Asymptotic mean stationarity: For the AMS property, we show that Assumption 2.2 holds: Let $X_{[-m,0]} \in B$ and $S_0 = 0$ have a zero measure under \bar{P} . Then, $\pi(B) = 0$. To show this, consider the contrapositive: If $\pi(B) > 0$, by the condition that all finite dimensional cylinder sets consisting of nonempty open sets have positive measure conditioned on any past event, it follows that for some $S_0 = s^*$ with positive measure under \bar{P} , there exists a positive probability event $X_{[0,m]} \in B$ so that $S_m = 0$. With

$$(4.2) \quad \begin{aligned} \bar{P}(X_{[-m,0]} \in B, S_0 = 0) &= \bar{P}(X_{[0,m]} \in B, S_m = 0) \\ &\geq \int_z \bar{P}(dz, s^*) \mathbb{P}(x_{[0,m]} \in B, S_m = 0 | z, s^*) > 0, \end{aligned}$$

it follows that the absolute continuity condition holds, and by Theorem 3.3, the AMS property.

Ergodicity: We can establish the uniqueness of an invariant probability measure through either irreducibility properties or the following argument. Consider the point $p_0 = \{m/2\}^{\mathbb{Z}^-} \times \{0\}$. We argue that this point is accessible. Recall that an open set in a product topology is a Borel set in the product space consisting of finitely many open sets with the rest being \mathbb{X} itself or an arbitrary union of such sets. Now, consider any $x_{(-\infty,0]}, s$. From this point, we will show that for every open neighborhood U of p_0 , there exists some $k > 0$ so that $P(X_{(-\infty,k]} \in U | x_{(-\infty,0]}, s) > 0$. For $x \in U$ for such U , $x_{(-\infty,0]} \in \prod_{l=-\infty}^0 A_l$ for finitely many non empty open sets which are not equal to \mathbb{X} and the rest being \mathbb{X} (see, e.g., [5, Theorem 2.4.4]). Let $-l$ be the largest index for which $A_{-l} \neq \mathbb{X}$. Hence, it is evident that $x_{(-\infty,l]}$ can take values in this open set for a given $x_{(-\infty,0]}$. We also need to ensure that S_l hits zero. To allow for this to happen, we further shift the process to the left: For any sufficiently small $\epsilon > 0$, identify a sequence of events from r to $r + l$ so that $S_{r+l} = 0$ for some $S_r = s$ when $|X_k - m/2| \leq \epsilon$ in this time interval. As a result, at time $r + l$ the state process hits 0 and the process $X_{(-\infty,r+l]}$ hits the open set with positive probability with $S_{r+l} = 0$. Finally, continuity holds due to the continuity of the noise process: The sets of points where continuity fails, $D = \{x : x = km, k \in \mathbb{Z}\}$, is a closed set and p_0 is outside this set. By Theorem 3.9, the process is ergodic. \square

4.1.2. Adaptive quantization of Goodman and Gersho. Consider the following update equations [8]:

$$(4.3) \quad \begin{aligned} V_t &= \Delta_t Q_1(X_t / \Delta_t), \\ \Delta_{t+1} &= \Delta_t Q_2\left(\frac{|X_t|}{\Delta_t}\right), \quad \Delta_0 = b. \end{aligned}$$

Here, Δ_t is the bin size of the uniform quantizer with a finite range and $|Q_1(\mathbb{R})| < \infty, |Q_2(\mathbb{R}_+)| < \infty$. V_t is the output which is to track the source process X_t . Suppose further that Q_2 is nondecreasing.

THEOREM 4.2. *Let X_t be a stationary and ergodic (nondeterministic) Gaussian sequence, $\zeta = \lim_{x \rightarrow \infty} Q_2(x) > 1$, $Q_2(0) = \lim_{x \downarrow 0} Q_2(x) < 1$, and $\log_2(Q_2(\cdot)) \in \mathbb{Q}$. Then, the system is stochastically stable. If, in addition, with $\{\alpha_1, \alpha_2, \dots, \alpha_L\}$ a set*

of pairwise relatively prime integers and $\log_2(Q_2(\cdot)) \in \{\alpha_k m\}$ for some $m \in \mathbb{Q}$, the process is AMS and, furthermore, ergodic.

Proof. Consider

$$\log_2(\Delta_{t+1}) = \log_2(\Delta_t) + \log_2\left(Q_2\left(\frac{|X_t|}{\Delta_t}\right)\right),$$

$\log_2(\Delta_t) - \log_2(\Delta_0) \in \mathbb{Q}$ for all t . Let $S_t = \log_2(\Delta_t)$. This sequence takes values in a countable set and satisfies

$$S_{t+n} - S_t = \sum_{k=t}^{t+n-1} \log_2\left(Q_2\left(\frac{|X_k|}{\Delta_k}\right)\right),$$

As in the proof of Theorem 4.1, [19, Theorem 4] shows that the coding scheme satisfies the condition [19, eq. (2.2)], which implies (4.1). By Theorem 2.6, the system is stochastically stable.

Asymptotic mean stationarity: Since $\{\alpha_k\}$ is a set of numbers that are relatively prime, \mathbb{S} consists of all integer multiples of m shifted by the initial value $\log_2(b)$. This follows from the property of relatively prime numbers due to Bézout’s lemma; see [32, Lemma 7.6.2]. The argument for the AMS property then follows as before through the absolute continuity condition: Any invariant measure is such that $\bar{P}(\cdot, s) \ll \bar{P}(\cdot, s')$ for all admissible s, s' and by Theorem 3.3, the result follows.

Ergodicity: In this case, the point $(\{0\}^{\mathbb{Z}^-}, \log_2(b))$ is accessible by the same arguments adopted in the proof of Theorem 4.1 and the steps leading to the AMS property above. By Theorem 3.9, the process is ergodic. \square

4.2. Stochastic networked control. We consider a stabilization problem in stochastic networked control where a linear system is controlled over a communication channel. We will study the approach in [31, 33] (see [32] for a detailed discussion). Consider the following control system, with U_t a control variable,

$$(4.4) \quad X_{t+1} = aX_t + bU_t + W_t,$$

where $|a| \geq 1$, W_t is i.i.d, admitting a probability measure ν which admits a density, positive everywhere and bounded. Furthermore, $E[|W_t|^{2+\zeta}] < \infty$ for some $\zeta > 0$.

In the application considered, a controller has access to quantized information from the state process. The quantization is described as follows. An adaptive quantizer has the following form with Q_K^Δ being a uniform quantizer with $K + 1$ bins and bin size Δ ; $Q_K^\Delta : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the following for $k = 1, 2, \dots, K$:

$$Q_K^\Delta(x) = \begin{cases} (k - \frac{1}{2}(K + 1))\Delta & \text{if } x \in [(k - 1 - \frac{1}{2}K)\Delta, (k - \frac{1}{2}K)\Delta), \\ \frac{1}{2}(K - 1)\Delta & \text{if } x = \frac{1}{2}K\Delta, \\ 0 & \text{if } x \notin [-\frac{1}{2}K\Delta, \frac{1}{2}K\Delta]. \end{cases}$$

With $K = \lceil |a| + \epsilon \rceil$, $R = \log_2(K + 1)$, let $R' = \log_2(K)$. We will consider the following coding and quantization update policy. For $t \geq 0$ and with $\Delta_0 > L$ for some $L \in \mathbb{R}_+$, and $\hat{x}_0 \in \mathbb{R}$, consider

$$U_t = -\frac{a}{b}\hat{X}_t, \quad \hat{X}_t = Q_K^\Delta(X_t),$$

$$\Delta_{t+1} = \Delta_t \bar{Q}\left(\left|\frac{X_t}{\Delta_t 2^{R'-1}}\right|, \Delta_t\right).$$

Suppose that with $\delta, \epsilon, \alpha > 0$ with $\alpha < 1$ and $L > 0$ the following hold:

$$\begin{aligned} \bar{Q}(x, \Delta) &= |a| + \delta && \text{if } |x| > 1, \\ \bar{Q}(x, \Delta) &= \alpha && \text{if } 0 \leq |x| \leq 1, \Delta \geq L, \\ \bar{Q}(x, \Delta) &= 1 && \text{if } 0 \leq |x| \leq 1, \Delta < L. \end{aligned}$$

THEOREM 4.3 (see [33, 31]). *Consider an adaptive quantizer applied to the linear control system described by (4.4). If the noiseless channel has capacity*

$$R > \log_2(\lceil |a| \rceil + 1),$$

and for the adaptive quantizer in (4.5), if the quantizer bin sizes are such that their (base 2) logarithms are integer multiples of some scalar s , and $\log_2(\bar{Q}(\cdot, \cdot))$ take values in integer multiples of s where the integers taken are relatively prime (that is they share no common divisors except for 1), then the process $\{(X_t, \Delta_t)\}$ is a positive (Harris) recurrent Markov chain (and has a unique invariant distribution).

In [33] it was shown that an m -small set (since a petite set in an irreducible and aperiodic Markov chain is m -small [21]) can be constructed so that return conditions are satisfied. Hence, the return time properties directly lead to a stability result. The small set discussion in [33] builds on the Markovian property and irreducibility and aperiodicity of the Markov chain, together with a *uniform countable additivity* condition from [29].

We can obtain the stability result through the analysis in this paper, without defining a small/petite set: One can view the system as $(\Delta_{t+1}, x_{t+1}) = F(\Delta_t, x_t, w_t)$, where the state is now $s_t := (\Delta_t, x_t)$ and the independence of w_t makes the process (Δ_t, x_t) Markov. Let the transition kernel be denoted by P . The finiteness of $\limsup_{t \rightarrow \infty} E[\Delta_t^2 + x_t^2]$ can be established by a Lyapunov analysis similar to [31, 16]. However, F here is not continuous in s_t . Nonetheless, the set of discontinuity is given by

$$D = \left\{ x, \Delta : \frac{x}{\Delta} \in \left\{ -\frac{K}{2}, \dots, \frac{K}{2} \right\}, \Delta \in \mathcal{N} \right\},$$

where \mathcal{N} is the set of admissible bin sizes which is a countable set. As a result, D is also countable and closed (since the elements are uniformly separated from each other). Furthermore, any weak limit of a converging sequence of expected occupational measures has zero measure on D , as can be deduced from the condition that every open set $D_\epsilon = \{x, \Delta : d((x, \Delta), D) < \epsilon\}$ is such that

$$\begin{aligned} v_{t_k} P(D_\epsilon) &= \sum_{\Delta} \int_z v_{t_k}(dz, \Delta) \sum_{\Delta_{t_k}} P(\Delta_{t_k} | x_{t_k-1} = z, \Delta_{t_k-1} = \Delta) \\ &\quad \times \sum_{m=-\frac{K}{2}}^{\frac{K}{2}} P(x_{t_k} \in [m\Delta_{t_k} - \epsilon, m\Delta_{t_k} + \epsilon] | x_{t_k-1} = z, \Delta_{t_k-1} = \Delta) \\ &\leq L_1 \epsilon \end{aligned}$$

for some $L_1 < \infty$ since $P(x_{t+1} \in dx | x, \Delta)$ has a density which is uniformly bounded for all z, Δ and the conditional probability $P(\Delta_{t_k} | x_{t_k-1} = z, \Delta_{t_k-1} = \Delta)$ has finite support. By Theorem 2.9, the result follows. Finally, ergodicity follows from the irreducibility of the Markov process.

5. Conclusion. In this paper, a method to verify stochastic stability, asymptotic mean stationary, and ergodicity properties of a class of non-Markovian stochastic

processes has been introduced. Applications to practical important feedback coding schemes and networked control have been investigated. Further applications in control of non-Markovian systems and stability of nonlinear filters are interesting research directions.

Appendix A. Ergodic theorems for Markov chains.

Suppose that $\{X_t\}_{t \geq 0}$ denote a discrete-time Markov chain with state space \mathbb{X} , a Polish space; its Borel σ -field is denoted by $\mathcal{B}(\mathbb{X})$, defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. The transition probability is denoted by P , so that for any $x, A \in \mathcal{B}(\mathbb{X})$, the probability of moving in one step from the state x to the set A is given by $\mathbf{P}(X_{t+1} \in A \mid x_t = x) = P(x, A)$. The n -step transitions are obtained via composition in the usual way, $\mathbf{P}(X_{t+n} \in A \mid X_t = x) = P^n(x, A)$ for any $n \geq 1$. The transition law acts on measurable functions $f: \mathbb{X} \rightarrow \mathbb{R}$ and measures μ on $\mathcal{B}(\mathbb{X})$ via $Pf(x) := \int_{\mathbb{X}} P(x, dy)f(y)$, $x \in \mathbb{X}$, and $\mu P(A) := \int_{\mathbb{X}} \mu(dx)P(x, A)$, $A \in \mathcal{B}(\mathbb{X})$. A probability measure π on $\mathcal{B}(\mathbb{X})$ is called invariant if $\pi P = \pi$, i.e.,

$$\int \pi(dx)P(x, A) = \pi(A), \quad A \in \mathcal{B}(\mathbb{X}).$$

For any initial probability measure ν , by the Ionescu Tulcea theorem [14], we can uniquely construct a stochastic process with transition law P , and satisfying $X_0 \sim \nu$. We let P_ν denote the resulting probability measure on the sample space $(\mathbb{X}, \mathcal{B}(\mathbb{X}))^{\mathbb{Z}^+}$, with the usual convention for $\nu = \delta_x$ when the initial state is $x \in \mathbb{X}$ in which case we write P_x for the resulting probability measure. Likewise, E_x denotes the expectation operator when the initial condition is given by $X_0 = x$. When $\nu = \pi$, the resulting process is stationary.

When an invariant probability measure is known to exist for a Markov chain, we state the following ergodicity results.

THEOREM A.1 (see [15, Theorems 2.3.4–2.3.5]). *Let \bar{P} be an invariant probability measure for a Markov process.*

(i) *(individual ergodic theorem). Let $X_0 = x$. For every $f \in L_1(\bar{P})$*

$$\frac{1}{N} E_x \left[\sum_{n=0}^{N-1} f(X_n) \right] \rightarrow f^*(x)$$

for all $x \in B_f$, where $\bar{P}(B_f) = 1$ (where B_f denotes that the set of convergence may depend on f) for some f^ .*

(ii) *(mean ergodic theorem). Furthermore, the convergence $\frac{1}{N} E_x[\sum_{n=0}^{N-1} f(X_n)] \rightarrow f^*(x)$ is in $L_1(\bar{P})$.*

THEOREM A.2 (see [15, Theorem 2.5.1]). *Let \bar{P} be an invariant probability measure for a Markov process. With $X_0 = x$, for every $f \in L_1(\bar{P})$,*

$$\frac{1}{N} \sum_{n=0}^{N-1} f(X_n) \rightarrow f^*(x)$$

for all $x \in B_f$, where $\bar{P}(B_f) = 1$ for some $f^(x)$ with*

$$\int \bar{P}(dx)f^*(x) = \int \bar{P}(dx)f(x).$$

One may state further refinements; see [15] for the locally compact case and [30] for the Polish state space case.

THEOREM A.3 (see [15, 30]). Let \bar{P} be an invariant probability measure for a Markov process.

- (i) (ergodic decomposition and weak convergence). For x , \bar{P} a.s., $\frac{1}{N}E_x[\sum_{t=0}^{N-1} 1_{\{x_n \in \cdot\}}] \rightarrow P_x(\cdot)$ weakly and \bar{P} is invariant for $P_x(\cdot)$ in the sense that

$$\bar{P}(B) = \int P_x(B)\bar{P}(dx).$$

- (ii) (convergence in total variation). For all $\mu \in \mathcal{P}(\mathbb{X}^{\mathbb{N}})$ which satisfies that $\mu \ll \bar{P}$ (that is, μ is absolutely continuous with respect to \bar{P}), there exists v^* such that

$$\left\| E_{\mu} \left[\frac{1}{N} \sum_{t=0}^{N-1} 1_{\{T^n X \in \cdot\}} \right] - v^*(\cdot) \right\|_{TV} \rightarrow 0.$$

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