Information Theoretic Study of the Signaling Problem in Decentralized Stabilization of Noisy Linear Systems

Serdar Yüksel [1] and Tamer Başar [2]

***Abstract*— We study the problem of distributed stabilization of linear systems over communication channels. Building on our earlier work, we adopt an information theoretic look at the signaling problem when the system and observations are noisy. We provide a lower bound on the average sum-rate, which is tight when the system noise is absent. We further show that when the system and observations are noisy, the signaling process involves coding over an unknown channel with unequal side information between the stations, and as such its construction is fairly complicated. This leads to new insights on designing distributed controllers connected over channels.**

I. Introduction

The use of digital and wireless channels such as the Internet or bus lines (as in a Controller Area Network (CAN)) in control systems has become common place, in particular in the context of distributed systems. Some typical areas of applications include environmental detection, detection of high-way congestion, unmanned vehicles, surveillance and rescue operations, as well as problems in formation control and aerospace applications; see for instance [12], [15]. The design and synthesis of control policies and algorithms in such systems connected over channels require an understanding on the quantitative value of partial information available at the controllers with regard to system performance.

Information theory provides a quantitative meaning to the value of a single bit, a meaning which however is only operational in the infinite-block case, except for special channels, such as Gaussian channels. Nonetheless, information theory provides limits to what is possible to transmit, and surprisingly perhaps, these bounds are closely attainable in certain settings, even in non-Gaussian contexts.

Distributed systems are especially challenging since the information structure can fall into the categories such that the optimization problem can be non-tractable. The so-called non-classical information structure is an example of such cases. However, there are instances when the non-classical information patterns do lead to computable solutions.

In this paper, we study the communication rate requirements in a class of decentralized systems, where two distributed stations attempt to stabilize a plant. We make the problem formulation precise in the following.

[1]S. Yüksel was with the department of Electrical Engineering, Yale University. He is now with the Mathematics and Engineering Program, Department of Mathematics and Statistics, Queen's University, Kingston, ON K7L 3N6, Canada. E-mail:yuksel@mast.queensu.ca. Research supported by National Science Foundation under Grant ECCS-0547199 and a Queen's University start-up grant.

[2]Coordinated Science Laboratory, University of Illinois, 1308 West Main Street, Urbana, IL 61801-2307 USA. E-mail: tbasar@decision.csl.uiuc.edu. Research supported in part by the NSF Grant CCR 00-85917 ITR.

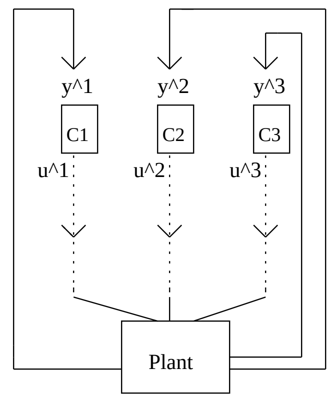

Fig. 1: A Multi-Controller System.

Consider a class of multi-station n-dimensional discrete-time, two controller, LTI systems

$$x(t+1) = Ax(t) + B^1u^1(t) + B^2u^2(t) + w(t)\,,$$
$$y^i(t) = C^ix(t) + v^i(t)\,,\ \ i = 1,2, \qquad (1)$$

where $(A, [B^1|B^2])$ is controllable and $(A, [(C^1)^T|(C^2)^T]^T)$ is observable, but the individual pairs may not enjoy this, such that (A, B^i) may not be controllable or (A, C^i) may not be observable, $i = 1, 2$. Here, $x(t) \in R^n$ is the state of the system, $u^i(t) \in R^{m_i}$ is the control applied by station i, and $y^i(t) \in R^{p_i}$ is the observation available at station i at time t. Here $\{w(t), v^i(t)\}$ are disturbance processes with arbitrary distributions, whose components are i.i.d random variables. We assume that the initial state x_0 is a random vector with a continuous probability density function, with a compact support set.

The control signals $u^i, i = 1, 2$, are coded and decoded over discrete noiseless channels with finite capacity. Hence, the applied control and transmitted messages follow a coding, binary representation, and a decoding process. We assume fixed-rate encoding, that is, the rate is defined as the (base-2) logarithm of the number of symbols to be transmitted: The coder maps the local observations up to time t to $\{1, 2, \ldots, W^i_t\}$, which is the quantizer codebook at station i at time t. Hence, at each time t, station i sends $\log_2(W^i_t)$ bits over the channel to the plant. Now, the information available at the plant is as follows. The plant knows the codebooks used by each controller, to be able to perform decoding. The decoder output at the plant with regard to the information received from station i at time t, which we again denote as $u^i_t \in R^{m_i}$, is generated through a (memoryless) mapping from $\{1, 2, \ldots, W^i_t\}$ to R^{m_i}. In addition to the quantizer policy and codebook, the plant also knows when the controllers at each station choose to signal information or to apply control. This is needed to ensure that the plant is capable of negating the effect of signaling.

This paper considers multi-controller systems as opposed to multi-sensor systems. One important difference between the decentralized multi-controller systems and and the multi-sensor systems with a centralized controller is the following: In a multi-sensor structure, there exists a centralized controller which assembles the observations from multiple sensors, generates an estimate (as in a fusion center) and computes the control. However, in a multi-controller setup, there is no centralized decoder at the plant. This is due to the fact that in a realistic scenario, the plant should be merely acting on the control signals received, for otherwise there would not be any need for the transmission of control over a communication channel. It should be observed, however, that the plant can still have local feedback control, and the discussion here is with regard to the control signals sent over to a remote location. If the plant is able to do filtering with regard to the controller actions, then the results with regard to the multi-sensor setup will be applicable. As an example, consider a robot vehicle which is being remotely controlled. The remote controller can develop an estimate on the vehicle's position, using the system dynamics, past received observations and the previously transmitted control signals. However, the vehicle should be designed so that it can act on any of the commands generated by the remote controllers, such as reducing acceleration, changing direction and so forth, even when the commands are random and hence unpredictable.

It should be noted that, in our framework the plant is not allowed to make joint decoding, for the reasons in [1]. Otherwise, the problem reduces to the analysis of multi-sensor systems as becomes an information relaying problem as was explicitly discussed in [13].

Let $\mathcal{R}$ denote the set of average (over time horizon) rates on 2 channels which lead to (decentralized) stabilization:

$$\begin{aligned}\mathcal{R} \quad &= \{R^1, R^2 : \exists \{u^1_{[0,\infty)}, u^2_{[0,\infty)}\} \\ &G < \infty, \limsup_{T\to\infty} E[|x(T)|^2] \leq G \quad \},\end{aligned}$$

where $R^i = \lim_{T\to\infty} \frac{1}{T}\sum_{k=1}^{T} \log_2(W_t^i)$, where W_t^i is the number of symbols transmitted by station i at time t, $i = 1, 2$. We seek to obtain bounds on the minimum achievable average sum rate $\min_{\mathcal{R}} \sum_{i=1}^{2} R^i$, such that decentralized stabilization is possible. $\diamond$

Literature Review

Decentralized stabilization has attracted considerable interest in the literature [10], [8]. One of the accomplishments in this domain is the introduction of *decentralized fixed modes* [6] and graph-theoretic characterization of stability [8] (see also [10]).

With regard to communication theoretic issues, most of the efforts in the literature assume either the multisensor structure or the multicontroller structure. For the multisensor structure, due to the assumption of a centralized decoder, one can use Slepian-Wolf coding theorem to arrive at the rate requirements. Reference [2] provides such a treatment of distributed control, and shows that the minimum rate required for stability of multi-sensor systems with a centralized controller is the same as the rate required in the centralized case. Reference [19] studies the rate-requirements for multi-sensor systems, and provides constructions. Reference [13] considers the optimal binning constructions for multi-sensor systems. [17] studies multi-sensor systems over erasure channels with feedback. Reference [4] carries out a sufficient rate analysis in a multicontroller setting, where the open-loop state dynamics are decoupled, and [18] studies the rate requirements for multi-controller systems when the observations are noisy. Reference [14] considers multi-controller systems with time-invariant policies where the sensors use state feedback. Reference [16] considers stabilizing coding and sensing policies in a distributed setting.

An earlier work [1] studied the minimum sum-rate required for the stabilizability of noiseless multi-controller systems. This paper builds on the approach originated in [1]. The contribution of this paper is to show that the result obtained in [1] provides a bound on the noisy case, which is, of course, not a surprising result. Furthermore, this paper shows that when noise is present in the system, the bound obtained is not tight. A discussion on signaling when system and observation noise is present suggests design schemes for distributed control over noisy channels.

II. Decentralized Stabilizability and a New Canonical Form

The discussion in this section follows [1]. In the following, we introduce some additional notation, and revisit decentralized stabilizability and connectivity.

We denote the controllability matrix for station i by $\mathcal{C}^i$, where

$$\mathcal{C}^i = \left[B^i | AB^i | \dots | A^{n-1}B^i\right],$$

and the observability matrix for station j by $\mathcal{O}^j$, which is given by

$$\mathcal{O}^j = \left[(C^j)^T | (C^jA)^T | \dots | (C^jA^{n-1})^T\right]^T.$$

We let N^i denote the unobservable subspace of station i, and K^i denote its controllable subspace. In other words, N^i is the null-space of $\mathcal{O}^i$, and K^i is the range space of $\mathcal{C}^i$. We define O^i to be the subspace orthogonal to N^i, and call it the observable subspace by a possible abuse of terminology. We call L^i the uncontrollable subspace of the ith controller.

We define a *mode* of a linear system as an eigenvector corresponding to an (open-loop) eigenvalue. The set of unstable modes is the set of eigenvectors of the system matrix corresponding to unstable eigenvalues. These notions naturally extend to generalized eigenvectors and to complex eigenvalues. When the systems are converted to controllable canonical forms, the *modes* are preserved; however, the states might be subject to a linear transformation.

We next review the notion of connectivity: If $N^j \not\supset K^i$, then station i can affect the observations of station j, and thus communicate to station j via control [9], which we capture through the notation $i \to j$. In this case, station i is said to be *connected* to station j. If every station is connected to every other station, possibly through other stations, the system is said to be strongly connected.

Through communication via the plant, the controllable subspace can be expanded and the unobservable subspace can be shrunk [9]. The following result follows from [8]: *Decentralized stabilization in a multi-controller setting is possible, if the system is jointly controllable, jointly observable, and strongly connected. Such a stabilization is in general possible through time-varying controls.*

We refer the reader to [10] for conditions of decentralized stabilization with a more restrictive set of controllers (such as, time-invariant, output feedback controllers, for which the absence of decentralized fixed modes is required in addition to joint controllability, observability, and strong connectivity). Hereafter, while performing the rate analysis, we will assume that in the absence of any structural restriction on the class of controllers, the system is decentrally stabilizable in the sense of [8].

A linear system can be expressed in a controllable canonical form such that $x = [x^1|x^2]$ and $\{x^1\} \subset K^1$. This can be achieved via a transformation matrix P such that the first n_1 columns of P^{-1} are the linearly independent columns in K^1. Due to the joint controllability assumption, it must be that $\{x^2\} \subset K^2$. The rest of the matrix P^{-1} can consist of linearly independent columns so long as the matrix P is full-rank ([20] pp.163). In the following we partition the state space: $x = [x_1|x_2]$, where

$$x_1 = [x^1, x^3], \quad x_2 = [x^2, x^4]$$

so that $\{x^1\} \subset K^1 \cap O^1$, $\{x^2\} \subset K^2 \cap O^2$,$\{x^3\} \subset K^1, \{x^3\} \subset O^1 \cup O^2$, $\{x^4\} \subset K^2, \{x^4\} \subset O^1 \cup O^2$. We obtain P^{-1} as a matrix whose first $\dim(\mathrm{x}^1)$ columns span $K^1 \cap O^1$, the rest is arbitrary so long as P is full-rank. In this case, we obtain Λ_1 as the upper-triangular of PAP^{-1}, as a matrix in the controllable mode expansion. In this case, the transformed system takes an upper-block triangular form. We apply the same approach for the modes in $K^2 \cap O^2$. Finally, once the dimension of the system is reduced, the same form is applied for the remaining blocks. The associated matrices in the block-diagonal can be represented as Λ_i. Hence, successively an upper triangular system matrix can be generated. We analyze the system starting from the lower block of the block-diagonal form.

We now provide a lower bound on the average sum rate. In essence, the following was proven in [1]. However, due to the presence of noise, there are a few technical steps and we will briefly discuss those in the development.

Theorem 2.1: Assume that unrestricted decentralized stabilization is possible, that is, the system is jointly controllable, jointly observable, and strongly connected. Suppose the the modes are ordered as $\{n_1, n_2, \ldots, n_n\}$, and $\{x^{n_1}, x^{n_2}, \ldots, x^{n_n}\}$ denote the ordered states. Let $\mathcal{N}$ be the set of such orderings, and for $i \geq 1$, $M_i :=$span$(x^{n_1}, x^{n_2}, \ldots, x^{n_i})$. Then, a lower bound on the sum rate required, between the controllers and the plant for stabilizability is given by

$$\min_{\{n_1,n_2,\ldots,n_n\} \in \mathcal{N}} \left\{ \sum_{|\lambda_i|>1} (\eta_{M_i} + 1) \bigg(\log_2(|\lambda_i|) \bigg) \right\}, \qquad (2)$$

where

$$\eta_i = \begin{cases} 1 & \text{if } \{x^i\} \nsubseteq (O^m \cup M_i) \cap (K^m \cup M_i), m = 1,2 \\ 0 & \text{else} \end{cases}$$

The following example, taken from [1], helps to illustrate the preceding result. Consider the system:

$$\begin{aligned} x_{t+1} &= Ax_t + B^1 u_t^1 + B^2 u_t^2 + w(t)\,, \ t \geq 0, \\ y_t^i &= C^i x_t + v^i(t)\,, \ \ i = 1,2 \end{aligned} \qquad (3)$$

with

$$A = \mathrm{diag}(2,2),$$

$$B^1 = [0 \quad 1]^T, \ C^1 = [1 \quad 1],$$

$$B^2 = [1 \quad 0]^T, \ C^2 = [1 \quad -1].$$

Here, neither of the stations can recover the modes of the system independently; however the system is decentrally stabilizable (even by linear policies). If the system were centralized, the average rate needed would be 2 bits. In the decentralized case, however, a lower bound on the average sum rate under the information structure for the controllers and the plant is 3 bits.

Proof: The rate of the signals from the controller to the plant is lower bounded by the entropy of the control variables, which leads to the mutual information being a lower bound on the rate. Let R_1 denote the average sum rate needed to stabilize x^1. Likewise let R_2 denote the average sum rate to stabilize x^2. Likewise R_3 and R_4 can be defined, with an additional difference; for the latter two sets of modes, it might be that signaling can be required, hence the rate will have contributions from both of the controllers. It now follows from the standard information theoretic arguments that the average rate of transmission from controller 1 to stabilize x^1 satisfies:

$$R^1 \geq \liminf_{t\to\infty} \frac{1}{t} I(x_0^1; u_{[0,t]}^1), \qquad (4)$$

where $u_{[0,t]}^1 = \{u_0^1, u_2^1, \ldots, u_t^1\}$. Likewise u^2, u^3, u^4 can be studied, with u_t^k denoting the control applied to act on states x^k at time t. Let Λ_1 be the controllable canonical output matrix for modes in the space spanned by x^1. We have that x_t^1 can be written as

$$\Lambda_1^t x_0^1 + f(x_0^2, x_0^3, x_0^4) + g(w_{[0,t-1]}) + \sum_{j=1}^{t-1} \Lambda_1^{t-j} B^1 u_t^1,$$

where $g(w_{[0,t-1]})$ is the upper $\dim(x^1)$ rows of $\sum_{j=1}^{t-1} \tilde{A}^{t-j} P w_t$, where $\tilde{A}$ is the transformed matrix, which is upper block triangular. Likewise $f(x_0^2, x_0^3, x_0^4)$ can be computed using the upper-block-triangular form of $\tilde{A}$. We have

$$\begin{aligned}
&h(x_t^1)\\
&= h(\Lambda_1^t x_0^1 + f(x_0^2, x_0^3, x_0^4) + g(w_{[0,t-1]})\\
&+ \sum_{j=1}^{t-1} \Lambda_1^{t-j} B^1 u_t^1) && (5)\\
&\geq h(\Lambda_1^t x_0^1 + f(x_0^2, x_0^3, x_0^4) + g(w_{[0,t-1]})\\
&+ \sum_j \Lambda_1^{t-j} B^1 u_t^1 | w_{[0,t-1]}) && (6)\\
&= h(\Lambda_1^t x_0^1 + f(x_0^2, x_0^3, x_0^4) + \sum_j \Lambda_1^{t-j} B^1 u_t^1) && (7)\\
&\geq h(\Lambda_1^t x_0^1 + f(.,.,.) + \sum_j \Lambda_1^{t-j} B^1 u_t^1 | x_0^2, x_0^3, x_0^4) && (8)\\
&= h(\Lambda_1^t x_0^1 + \sum_j \Lambda_1^{t-j} B^1 u_t^1) && (9)\\
&\geq h(\Lambda_1^t x_0^1 + \sum_j \Lambda_1^{t-j} B^1 u_t^1 | \sum_j \Lambda_1^{t-j} B^1 u_t^1) && (10)\\
&= h(\Lambda_1^t x_0^i | \sum_j \Lambda_1^{t-j} B^1 u_t^1) && (11)\\
&\geq t \log_2(|\Lambda_1|) + h(x_0^1 | \sum_j \Lambda_1^{t-j} B^1 u_t^1) && (12)\\
&\geq t \log_2(|\Lambda_1|) + h(x_0^1 | u_{[0,t]}^1). && (13)
\end{aligned}$$

(5) follows from the evolution of the states, and (6) from the fact that conditioning does not increase the entropy. Equation (7) follows from the independence of the disturbance, (8) from conditioning, and (9) from the fact that the intersection of the subspaces is zero. (10) follows from the properties of conditioning, (11) from shifting of random variables, and (12) and (13) by the properties of conditional entropy.

It follows by rearranging terms, that

$$h(x_0^1) - h(x_0^1 | u_{[0,t]}^1) \geq h(x_0^1) + t \log_2(|\Lambda_1|) - h(x_t^1)\,.$$

Now, for a sequence of scalar continuous random variables $\{v_t, t = 1, 2, \dots\}$ having a finite variance, $\lim_{t\to\infty} E[v_t^2] < \infty$, the (differential) entropy has the property that $\limsup_{t\to\infty} h(v_t) < \infty$. Hence,

$$\liminf_{t\to\infty} \frac{1}{t} I(x_0^1; u_{[0,t]}^1) \geq \log_2(|\Lambda_1|).$$

It should be observed that $u_{[0,t]}^2$ is a causal function of the information available, $y_{[0,t]}^2$, and thus by the data processing inequality, it follows that

$$I(x_0^1; u_{[0,t]}^2) \leq I(x_0^1; y_{[0,t]}^1, y_{[0,t]}^2),$$

and

$$\liminf_{t\to\infty} \frac{1}{t} I(x_0^1; y_{[0,t]}^1, y_{[0,t]}^2) \geq \log_2(|\Lambda_1|)$$

Hence, the end to end rate satisfies

$$R_1 \geq \liminf_{t\to\infty} \frac{1}{t} I(x_0^1; y_{[0,t]}^1, y_{[0,t]}^2)$$

Further, it follows that $\limsup_{t\to\infty} \frac{1}{t} I(x_0^1; y_{[0,t]}^1)$ is finite and the information that needs to be signaled from the second controller to the first one satisfies

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{1}{t} I(x_0^1; y_{[0,t]}^2 | y_{[0,t]}^1)\\
&\geq \log_2(|\Lambda_1|) - \limsup_{t\to\infty} \frac{1}{t} I(x_0^1; y_{[0,t]}^1)\\
&\geq \log_2(|\Lambda_1|) - \limsup_{t\to\infty} \frac{1}{t} I(x_0^1; y_{[0,t]}^1)
\end{aligned}$$

Since, $\limsup_{t\to\infty} \frac{1}{t} I(x_0^1; y_{[0,t]}^1)) = \log_2(|\Lambda|_1)$, the amount that needs to be signaled is zero. A similar discussion follows for the modes x^2. Next, we work on the lower block of the controllable canonical form. Here, we work with x^2 and carry out the analysis above. We are then left with x^3 and x^4. Now, we apply another canonical transformation such that only x^3 is controllable. For x^3 we have, with $x^3 \subset O^1 \cup O^2$, the following.

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{1}{t} I(x_0^3; y_{[0,t]}^2 | y_{[0,t]}^1, x_0^4)\\
&\qquad \geq \log_2(|\Lambda_3|) - \limsup_{t\to\infty} \frac{1}{t} I(x_0^3; y_{[0,t]}^1 | x_0^4)
\end{aligned}$$

This is the amount of information that needs to be signaled, hence R_3 will consist of two terms; the control signal sent from station 1, as well as the signaling outputs from station 2. Finally, we consider $\{x^4\}$. For this, one again applies a canonical transformation such that only these modes are controllable. Using the observation that the initial values of the mode states are independent, the result follows. The order of $\{x^3\}$ and $\{x^4\}$, and the modes belonging to these, can be adjusted so as to pick the lower value. ⋄

Remark 1: In the special case where the modes are completely decoupled, the result reduces to the centralized case. In case modes are coupled, the controllable subspaces can be partitioned, and an analysis on each of the subspaces can be carried over. The above result is tight when the channels are noiseless [1]. ⋄

Remark 2: The analysis is applicable to more than two controllers. In this case, consider the following setup: If there is only one controller, controller k, with a controllable (A, B^k) pair, and if all other controllers have null controllable subspaces, then the problem reduces to a version of the multi-sensor problem, with the additional control signal to be sent to the plant from controller k. ⋄

III. Decentralized Stabilizability in the Presence of Noise and Signaling over a Noisy Plant

We first have the following negative result.

Theorem 3.1: When the system noise ($w(t)$) and the observation noise ($v(t)$) are present in the system, the lower bound in Theorem 2.1 is not achievable.

Toward the proof, we first present some preliminary results. Consider the configuration of Figure 2.

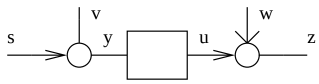

Fig. 2: Cascade of Channels.

We have that

$$\begin{aligned} I(s;y) - I(s;z) \quad &= -h(s|y) + h(s|z) \\ &= -h(s|y,z) + h(s|z) = I(s;y|z) \\ &> 0 \end{aligned} \tag{14}$$

Hence, it follows that the end to end mutual information is less than the mutual information in the first channel, unless

$$x \leftrightarrow z \leftrightarrow y$$

forms a Markov chain, implying that there is no information loss with regard to the message at the second channel. In the following we present some insight into this result. First consider the following lemma [7]:

Lemma 3.1: Consider the scheme in Figure 2, with $E[u^2] \leq P$ and v, w Gaussian. It follows that

$$\begin{aligned} E[(s - E[s|z])^2] &= E[(s - E[s|y])^2] \\ &\quad + E[(E[s|y] - E[E[s|y]|z])^2], \end{aligned} \tag{15}$$

and each of these terms can be minimized by using linear policies.

Lemma 3.2: Consider the scheme in Figure 2, with $E[u^2] \leq P$ and v, w Gaussian. Then $I(s;z)$ is maximized with $u = \alpha y$ such that $E[u^2] = P$.

Theorem 3.2: The mutual information between s and z satisfies:

$$I(s;z) < \min(I(s;y), I(y;z)) < \min(C_1, C_2)$$

Hence, there is a leak in the end to end information transmission when the channels are noisy. As such, there is a loss in the information transfer due to the noisy nature of the system. This is despite the fact that the information theoretic capacity of a block-code is still equal to $C = \min(C_1, C_2)$.

Signaling through the plant is equivalent to coding over an induced noisy channel. To develop some insight into the problem of signaling in noisy systems, let us consider the two controller system with

$$A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$$

$$C^1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad C^2 = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

$$B^1 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T \quad B^2 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T.$$

Furthermore, suppose the control signals are power constrained such that $E[(u_i(t))^2] \leq P_i, \quad i = 1, 2$ for all time stages.

Theorem 3.3: Let $\sigma^2_{x_i}(t) = E[x_i^2(t)]$, for $i = 1, 2$. For the stability of the system described, the set of P_1, P_2 values such that the joint non-linear equations below converge to a finite value is stabilizing.

$$\begin{aligned} \sigma^2_{x_1}(t+1) &= \frac{a_1^4 \sigma^2_{x_1}(t)}{(1 + \frac{P_1}{(\sigma^2_{w_2} + \sigma^2_{v_2} + a_2^2 \frac{\sigma^2_{x_2}(t)\sigma^2_{v_2}}{\sigma^2_{x_2}(t) + \sigma^2_{v_2}})})} \\ &\quad + (1 + a_1^2)\sigma^2_{w_1}, \\ \sigma^2_{x_2}(t+1) &= \frac{a_2^4 \sigma^2_{x_2}(t)}{(1 + \frac{P_2}{(\sigma^2_{w_1} + \sigma^2_{v_1} + a_1^2 \frac{\sigma^2_{x_1}(t)\sigma^2_{v_1}}{\sigma^2_{x_1}(t) + \sigma^2_{v_1}})})} \\ &\quad + (1 + a_2^2)\sigma^2_{w_2} \end{aligned}$$

Proof: Suppose the controllers signal their associated observed states, and apply their controls every other time stage. We consider the case where both signaling and state estimation processes are performed in a memoryless fashion, and also that the plant undoes the effects of signaling at every period of signaling.

Hence, the control acts on a lifted state, whereas each station keeps observing the channel state value with each time stage. We have the observation at station 1 as

$$y_1(t+1) = a_1 x_1(t) + w(t) + u_2(t) + v_1(t+1),$$

where $x_1(t)$ is unavailable at the station 1. Let $I_i(t)$ denote the information available at station i. Hence,

$$\begin{aligned} \bar{y}_1(t+1) &= a_1(x_1(t) - E[x_1(t)|I_1(t)]) \\ &\quad + w(t) + v_1(t+1) + u_2(t), \end{aligned} \tag{17}$$

where $\bar{y}_1(t+1) = y_1(t+1) - aE[x_1(t)|I_1(t)]$. In this case, the transmitter sends a message over an induced channel with disturbance values $a_1(x_1(t) - E[x_1(t)|I_1(t)]) + w(t) + v_1(t+1)$, which is a Gaussian random variable if linear policies are adopted.

The capacity of this channel, C_1, is

$$\frac{1}{2}\log_2(1 + \frac{P_1}{(\sigma^2_{w_2} + \sigma^2_{v_2} + a_1^2 E[(x_2(t) - E[x_2(t)|I_2(t)])^2]}),$$

and likewise, C_2 can be expressed as

$$\frac{1}{2}\log_2(1 + \frac{P_2}{(\sigma^2_{w_1} + \sigma^2_{v_1} + a_1^2 E[(x_1(t) - E[x_1(t)|I_1(t)])^2]}).$$

It should be observed that, the induced channel has a state estimation error component. This is in contrast with the analysis for the case with noiseless observations. We have, by the optimal memoryless mean square estimates:

$$\begin{aligned} E[(x_1(t) - E[x_1(t)|I_2(t)])^2] &= E[x_1(t)^2]e^{-2C_1} \\ E[(x_2(t) - E[x_2(t)|I_1(t)])^2] &= E[x_2(t)^2]e^{-2C_2} \end{aligned}$$

But, the steady state itself is a function of the control applied. In this case, the steady state value satisfies $x_1(t+2) = a_1^2 x_1(t) + u_2(t) + a_1 w(t) + w(t+1)$, and since the control is applied in every two time stages, it follows that

$$\begin{aligned} x_1(t+2) = (a_1)^2(x_1(t) - E[x_1(t)|I_2(t)]) + a_1 w(t) \\ + w(t+1) \end{aligned}$$

and

$$E[(x_1(t+1))^2] = a_1^2 E[(x_1(t) - E[x_1(t)|I_2(t)])^2] + (1+a_1^2)\sigma_{w_1}^2 \quad (18)$$

Hence, we have a non-linear dynamical system for $E[(x_i(t))^2]$:

$$E[x_1^2(t+1)] = a_1^4 E[x_1^2(t)] e^{-2C_1} + (1+a_1^2)\sigma_{w_1}^2$$
$$E[x_2^2(t+1)] = a_2^4 E[x_2^2(t)] e^{-2C_2} + (1+a_2^2)\sigma_{w_2}^2$$

Then,

$$\begin{aligned} & E[(x_1(t+1))^2] \\ = & \frac{a_1^4 E[x_1(t)^2]}{(1 + \frac{P_1}{(\sigma_{w_2}^2 + \sigma_{v_2}^2 + a_1^2 E[(x_2(t) - E[x_2(t)|I_2(t)])^2]})} \\ + & (1+a_1^2)\sigma_{w_1}^2, \\ & E[(x_2(t+1))^2] \\ = & \frac{a_2^4 E[x_2(t)^2]}{(1 + \frac{P_2}{(\sigma_{w_1}^2 + \sigma_{v_1}^2 + a_1^2 E[(x_1(t) - E[x_1(t)|I_1(t)])^2]})} \\ + & (1+a_2^2)\sigma_{w_2}^2 \end{aligned} \quad (19)$$

Finally, observe that

$$E[x_2(t)|x_2(t)+v_2(t)] = \frac{E[x_2^2(t)]}{E[x_2^2(t)] + \sigma_{v_2}^2}(x_2(t)+v_2(t))$$

It follows that

$$\begin{aligned} & E[(x_2(t) - E[x_2(t)|x_2(t)+v_2(t)])^2] \\ & = E[x_2^2(t)] - (E[x_2^2(t)])^2/(E[x_2^2(t)] + \sigma_{v_2}^2) \\ & = \frac{E[x_2^2(t)]\sigma_{v_2}^2}{E[x_2^2(t)] + \sigma_{v_2}^2} \end{aligned} \quad (20)$$

This concludes the proof. $\diamond$

Hence, we observe that the signaling process over a plant with unknown state values can be modeled as coding over an unknown channel, with the further property that the side information available at the encoder and the decoder are unequal, since the encoder and the decoder might have different information with regard to the state of the plant. As we observed, in this case the new channel is a noisy channel, and as such the lower bound obtained in Theorem 2.1 is no longer tight.

IV. Concluding Remarks

In this paper, we studied the problem of signaling and control for decentralized stabilizability. We obtained a lower bound building on the analysis in [1], which is not tight when noise is present. We observed that when noise is present the signaling problem becomes a complicated coding process over a channel with unequal channel side information at the transmitter and the receiver. The results here are applicable to more than two controllers following the discussion in [1]. In this case, however, the signaling problem becomes more tedious to investigate. One conclusion is that the analysis performed for noiseless systems suggests a time-varying, switching or a periodic control scheme, which can also be applicable to noisy systems. One research direction would be to extend the results of [22].

V. Acknowledgements

S. Yüksel acknowledges the invaluable discussions with Sekhar Tatikonda, Yale University, and for being his host while some of this work was performed.

References

[1] S. Yüksel and T. Başar, "Optimal signaling policies for decentralized multi-controller stabilizability over communication channels", *IEEE Trans. Automatic Control*, to appear September 2007.

[2] S. Tatikonda, "Some scaling properties of large distributed control systems," *Proc. 42nd IEEE Conf. Decision and Control*, December 2003, Maui, Hawaii, pp. 3142 - 3147

[3] S. Tatikonda, "Control Under Communication Constraints," PhD dissertation, Massachuetsess Institute of Technology, Cambridge, MA 2000.

[4] G. N. Nair, R. J. Evans and P. E. Caines, "Stabilising decentralised linear systems under data rate constraints", in *Proc. 43rd IEEE Conf. Decision and Control*, pp. 3992 - 3997, The Bahamas, December 2004

[5] A. S. Matveev and A. V. Savkin "Stabilization of multisensor networked control systems with communication constraints," in *Proc. Asian Control Conference*, Melbourne, Australia, July 2004.

[6] S. Wang and E.J. Davison, "On the stabilization of decentralized control. systems, *IEEE Trans. Automatic Control*, vol. 18, pp. 473-478, Oct. 1973.

[7] R. Bansal and T. Başar, "Solutions to a class of linear-quadratic-Gaussian LQG stochastic team problems with nonclassical information," *Systems and Control Letters*, 9:125–130, 1987.

[8] B. D. O. Anderson and J. B. Moore, "Time-varying feedback decentralized control," *IEEE Trans. Automatic Control* vol.26, pp. 1133-1139, Oct. 1981.

[9] H. Kobayashi, H. Hanafusa and T. Yoshikawa, "Controllability under decentralized information structure," *IEEE Trans. Automatic Control* vol. 23, pp. 182-188, Apr. 1978.

[10] J. P. Corfmat and A. S. Morse, "Decentralized control of linear multivariable systems," *Automatica*, vol. 11, pp. 479-497, Sep. 1976.

[11] H. Kobayashi and T. Yoshikawa, "Graph theoretic approach to controllability and localizability of decentralized control systems," *IEEE Trans. Automatic Control* vol. 27, pp. 1096-1107, Oct. 1982.

[12] J. A. Fax and R. M. Murray, "Information flow and cooperative control of vehicle formations," *IEEE Trans. Automatic Control*, vol. 49, pp. 1465–1476, Sept. 2004.

[13] S. Yüksel and T. Başar, "On the absence of rate loss in decentralized sensor and controller structure for asymptotic stability,"*Proc. IEEE American Control Conf.*, Minneapolis, MN, June 2006.

[14] S. Yüksel and T. Başar, "Communication constraints for decentralized stabilizability with time-invariant policies," *IEEE Trans. Automatic Control*, 52: 1060-1066, June 2007.

[15] G. Inalhan, D. M. Stipanovic, and C. J. Tomlin, "Decentralized optimization with application ot multiple aircraft coordination, in *Proc. IEEE Conference on Decision and Control*, December 2002.

[16] C. Zhang, G. Dullerud, "Decentralized control with communication bandwidth constraints", to appear in *Proc. IEEE American Control Conf.*, New York, NY, June 2007.

[17] V. Gupta, N. Martins and J. S. Baras "Optimal output feedback control using two remote sensors over erasure channels", submitted to *IEEE Trans. Automatic Control*, 2007.

[18] G. N. Nair and J. Evans, "Cooperative networked stabilisability of linear systems with measurement noise", *Proc. 15th IEEE Mediterranean Conf. Control Automation*, Athens, Greece, June 2007.

[19] A. S. Matveev and A. V. Savkin, "Multirate stabilization of linear multiple sensor systems via limited capacity communication channels", *SIAM J. Control Optimization*, 28:129-137, 1982.

[20] C. Chen "Linear System Theory and Design", *Linear System Theory and Design*, 3rd edition, Oxford University Press, New York, NY, USA.

[21] S. Jiang and P. Voulgaris, "Design of distributed controllers with constrained and noisy links", *Int. J. of Robust and Nonlinear Control*, 16: 891-912, 2006

[22] O. C. Imer and T. Başar, "To measure or to control: Optimal control of LTI systems with scheduled measurements and controls," *Proc. IEEE American Control Conf.*, Minneapolis, MN, June 2006.