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Brief paper

On optimal policies for control and estimation over a Gaussian relay channel☆

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ABSTRACT

The problem of causal transmission of a memoryless Gaussian source over a two-hop memoryless Gaussian relay channel is considered. The source and the relay encoders have average transmit power constraints, and the performance criterion is mean-squared distortion. The main contribution of this paper is to show that unlike the case of a point-to-point scalar Gaussian channel, linear encoding schemes are not optimal over a two-hop relay channel in general, extending the sub-optimality results which are known for more than three hops. In some cases, simple three-level quantization policies employed at the source and at the relay can outperform the best linear policies. Further a lower bound on the distortion is derived and it is shown that the distortion bounds derived using cut-set arguments are not tight in general for sensor networks.

1. Introduction

Consider a physical phenomenon characterized by a sequence of independent and identically distributed real valued Gaussian random variables $\{X_n\}_{n\in\mathbb{Z}_+}$ having zero mean and variance σ_x^2, where n denotes a discrete time index. We wish to instantly communicate this physical phenomenon to a remote destination over a two-hop relay channel with as high fidelity as possible. The system model is illustrated in Fig. 1. According to the figure, at a discrete time $n \in \mathbb{Z}_+$ the source encoder $\mathcal{E}$ observes X_n and produces $S_{e,n} = f_{1,n}(\{X_i\}_{i=1}^n)$ suitable for transmission, where $f_{1,n} : \mathbb{R}^n \mapsto \mathbb{R}$ is a causal measurable mapping. The mapping $f_{1,n}$ has to satisfy the following average power constraint,

$$\mathbb{E}[S_{e,n}^2] \le P_S. \tag{1}$$

The signal $S_{e,n}$ is then observed in noise by the relay node $\mathcal{R}$ as $Y_n = S_{e,n} + Z_{r,n}$, where $\{Z_{r,n}\}_{n\in\mathbb{Z}_+}$ is a zero mean white Gaussian noise sequence of variance N_r. Since there is no direct link from the source encoder to the destination, we neglect transmission and processing delays at the relay, i.e., the relay node applies a causal mapping on the received signal $f_{2,n} : \mathbb{R}^n \mapsto \mathbb{R}$ to produce $S_{r,n} = f_{2,n}(\{Y_i\}_{i=1}^n)$ under the power constraint,

$$\mathbb{E}[S_{r,n}^2] \le P_R. \tag{2}$$

The signal $S_{r,n}$ is then transmitted over a Gaussian channel. Accordingly the destination node $\mathcal{D}$ receives $R_n = S_{r,n} + Z_{d,n}$, where $\{Z_{d,n}\}_{n\in\mathbb{Z}_+}$ is a zero mean white Gaussian noise sequence of variance N_d. Upon receiving R_n the decoder wishes to reconstruct the transmitted variable X_n by applying a mapping $g_n : \mathbb{R}^n \mapsto \mathbb{R}$ to produce $\hat{X}_n = g_n(\{R_i\}_{i=1}^n)$. We define the signal-to-noise ratios of $\mathcal{E}$–$\mathcal{R}$ and $\mathcal{R}$–$\mathcal{D}$ links as $\gamma_r := P_S/N_r$ and $\gamma_d := P_R/N_d$ respectively. The encoder, the relay, and the decoder are all causal and delay-free (zero delay). The objective is to choose the encoder, relay, and decoder mappings such that following distortion

$$D = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{E}[(X_n - \hat{X}_n)^2] \tag{3}$$

is minimized subject to the constraints in (1) and (2).

It is well-known that linear encoding is optimal for transmission of a Gaussian source over a point-to-point scalar Gaussian channel when the distortion measure is the mean squared error (Berger, 1971; Goblick, 1965). From Ayanoglu and Gray (1987), Bansal and Başar (1987), Dobrushin and Tsybakov (1962), Fine (1965) we know that linear policies are also optimal if the encoder observes a noisy version of a Gaussian source. Moreover in Gastpar (2008) Gastpar has shown that a linear (uncoded) scheme is even optimal in a simple Gaussian sensor network setting where each

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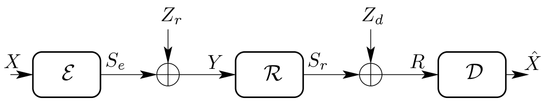

Fig. 1. System model.

sensor node observes a noisy version of a Gaussian source and all the sensor nodes simultaneously transmit over a multiple-access Gaussian channel.

In this paper, we show that linear policies are not optimal in general for the transmission of a Gaussian source over a Gaussian channel comprised of one or more relay nodes connected in cascade. A special case of this problem was studied by Lipsa and Martins in Lipsa and Martins (2011) where they provided counterexamples based on binary quantizers to show that linear policies are not optimal when the number of relays are greater than or equal to three. Further we discuss that linear encoding policies are person-by-person optimal[2] for a two-hop relay channel. However they do not guarantee global optimality because the given team problem is concave in the encoding policies, as observed in Yüksel and Linder (2011). We also derive a lower bound on distortion which is not tight in general.

This is a team decision problem under non-classical information structure (Ho, 1980). Some such problems are very difficult depending on the cost function and the information structure. The problem under study in this paper is a variation of the Witsenhausen's problem (Witsenhausen, 1968) with the addition of a relay encoder.

The problem of causal transmission over a two-hop relay channel is motivated by control applications, where the sensor measurements of a dynamical system are transmitted via a relay node to a remote controller. Control of linear systems over various types of relay channels has been studied in Kumar, Gupta, and Laneman (2010), Zaidi, Oechtering, and Skoglund (2010); Zaidi, Oechtering, Yüksel, and Skoglund (2011), where linear schemes are used to derive conditions on mean-square stability. In Karlsson and Skoglund (2010), non-linear policies are shown to outperform linear policies for instantaneous transmission of a Gaussian source over an orthogonal relay channel. A similar observation is made in the control context for parallel Gaussian channels (Andersson, Zaidi, Wernersson, & Skoglund, 2011; Yüksel & Tatikonda, 2007). The paper (Andersson et al., 2011) shows linear policies are not even person-by-person optimal for control over parallel Gaussian channels. However the problem under study in this paper is fundamentally different from the problems addressed in Andersson et al. (2011), Karlsson and Skoglund (2010), Yüksel and Tatikonda (2007) due to a different system model.

Notations: a sequence of variables is defined as $X^N := \{X_n\}_{n=1}^N$. The probability density function of a random variable X is denoted as $p(x)$ instead of $p_X(x)$ (we drop the subscript for ease of notation and its usage will be clear from the context). The function $Q(x) \triangleq \frac{1}{\sqrt{2\pi}} \int^{\infty}_{x} e^{-\frac{\tau^2}{2}} d\tau$ is the complement of the standard normal cumulative distribution function. The operator $\mathbb{E}[\cdot]$ is used for expectation.

2. Distortion lower bound

Consider the following series of inequalities:

$$
\begin{aligned}
\frac{1}{N}\sum_{n=1}^{N} I(X_n;\hat{X}_n) &\overset{(a)}{=} \frac{1}{N}\left(\sum_{n=1}^{N} H(X_n) - \sum_{n=1}^{N} H(X_n|\hat{X}_n)\right)\\
&\overset{(b)}{=} \frac{1}{N}\left(H(X^N) - \sum_{n=1}^{N} H(X_n|\hat{X}_n)\right)\\
&\overset{(c)}{\leq} \frac{1}{N}\left(H(X^N) - \sum_{n=1}^{N} H(X_n|\hat{X}^N, X^{n-1})\right)\\
&= \frac{1}{N}\left(H(X^N) - H(X^N|\hat{X}^N)\right) = \frac{1}{N} I(X^N;\hat{X}^N)\\
&\overset{(d)}{\leq} \frac{1}{N} I(S_e^N; R^N) \overset{(e)}{\leq} \frac{1}{N}\min\{I(S_e^N;Y^N), I(S_r^N;R^N)\}\\
&\overset{(f)}{\leq} \frac{1}{N}\min\left\{\sum_{n=1}^{N} I(S_{e,n};Y_n), \sum_{n=1}^{N} I(S_{r,n};R_n)\right\}\\
&\overset{(g)}{\leq} \frac{1}{2}\min\left\{\log\left(1+\frac{P_S}{N_r}\right), \log\left(1+\frac{P_R}{N_d}\right)\right\}, \qquad (4)
\end{aligned}
$$

where (a) follows from the definition of mutual information; (b) follows from the independence of the sequence X_n^N; (c) follows from the fact that conditioning reduces entropy; (d) and (e) follow from the data processing inequality (Cover & Thomas, 2006, Theorem 2.8.1) with Markov chain $X^N - S_e^N - R^N - \hat{X}^N$; (f) follows from the fact that the channels are memoryless and conditioning reduces entropy; and (g) follows from the fact that mutual information is maximized by Gaussian distribution among random variables with a given variance (Cover & Thomas, 2006, Theorem 8.6.5). Further consider the following inequalities:

$$
\begin{aligned}
&\frac{1}{N}\sum_{n=1}^{N} I(X_n;\hat{X}_n)\\
&\overset{(a)}{\geq} \frac{1}{2N}\sum_{n=1}^{N}\log\left(\frac{\sigma_x^2}{\mathbb{E}[(X_n-\hat{X}_n)^2]}\right)\\
&\overset{(b)}{\geq} \frac{1}{2}\log\left(\sigma_x^2\right) - \frac{1}{2}\log\left(\frac{1}{N}\sum_{n=1}^{N}\mathbb{E}\left[(X_n-\hat{X}_n)^2\right]\right), \qquad (5)
\end{aligned}
$$

where (a) follows from the rate distortion theorem for an i.i.d. Gaussian source (Cover & Thomas, 2006); and (b) follows from the concavity of the logarithm function and Jensen's inequality. From (4) and (5), we obtain the following lower bound:

$$
\begin{aligned}
D &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mathbb{E}\left[(X_n-\hat{X}_n)^2\right]\\
&\geq \sigma_x^2 \max\left\{\frac{N_r}{P_S+N_r}, \frac{N_d}{P_R+N_d}\right\}\\
&= \frac{\sigma_x^2}{1+\min\{\gamma_r,\gamma_d\}}. \qquad (6)
\end{aligned}
$$

Remark 1. The lower bound has been obtained without using causality constraints. Due to the two channel noise components (Z_r, Z_d) with non-zero variance, we have

$$
\max_{P_{S_e}} I(S_e;R) < \min\left\{\max_{P_{S_e}} I(S_e;Y); \max_{P_{S_r}} I(S_r;R)\right\}.
$$

As a result of this strict inequality, the bound (6) is not tight if we restrict the encoding policies to be memoryless.[3] This bound will be tight for memoryless policies when the variance of any of the two channel noise components approaches zero. This observation

[2] In a team decision problem, policies of decision makers are person-by-person optimal if there is no incentive for any decision maker to uni-laterally deviate from its policy given others' policies (Ho, 1980).

[3] A policy that only uses the current input at any time n.

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can also be made from Bansal and Başar (1989). In Yüksel and Tatikonda (2007, Theorem 3.5) the authors discussed that $I(S_e; R)$ is strictly lower than the capacity of a two-hop relay channel which follows from block coding arguments and cut-set bound. This tells us that the distortion bounds obtained using cut-set arguments are not tight in general for relay networks with memoryless policies.

3. Linear policies

In this section, we find the optimal linear policies and show that linear policies are person-by-person optimal. Moreover, it is shown that the person-by-person optimality of linear policies does not guarantee global optimality.

3.1. Optimal linear encoding

Typically when a source is memoryless and the encoders are causal, the optimal encoders are memoryless (Walrand & Varaiya, 1983). This can be easily verified by showing that if we transmit a linear combination of the current and the previous source observations, then the previous observations will only contribute to noise as the source is memoryless. We therefore restrict our study to memoryless linear policies, in the sense that the encoders merely transmit a scaled version of the received signal. That is, the source and the relay encoders transmit the following signals:

$$S_{e,n} = \sqrt{\frac{a_n}{\sigma_x^2}} X_n, \qquad S_{r,n} = \sqrt{\frac{b_n}{a_n + N_r}} Y_n,$$

where $a_n, b_n \in \mathbb{R}_+$ are time varying gain coefficients which are chosen such that the transmit power constraints in (1) and (2) are satisfied, i.e., $a_n \leq P_S$ and $b_n \leq P_R$. The decoder accordingly receives

$$R_n = \sqrt{\frac{a_n b_n}{\sigma_x^2 (a_n + N_r)}} X_n + \sqrt{\frac{b_n}{a_n + N_r}} Z_{r,n} + Z_{d,n},$$

and computes the MMSE estimate according to $\hat{X}_n = \mathbb{E}[X_n|R^n] = \mathbb{E}[X_n|R_n]$, where we have used the fact that the $\{R_n, R^{n-k}\}$ are mutually independent for all $k \neq n$. Since X_n is Gaussian, the distortion is given by

$$\mathbb{E}[(X_n - \hat{X}_n)^2] = \sigma_x^2 \left(1 - \frac{a_n b_n}{(a_n + N_r)(b_n + N_d)}\right),$$

$$\Rightarrow D_L = \lim_{N \to \infty} \frac{\sigma_x^2}{N} \sum_{n=1}^{N} \left(1 - \frac{a_n b_n}{(a_n + N_r)(b_n + N_d)}\right). \tag{7}$$

The optimal choice of the gain coefficients $0 < a_n \leq P_S$, $0 < b_n \leq P_R$, which minimizes (7) is $\{a_n^\star = P_S, b_n^\star = P_R\}$. This choice of the gain coefficients leads to

$$D_L^\star = \sigma_x^2 \left(1 - \frac{1}{(1 + \gamma_r)(1 + \gamma_d)}\right). \tag{8}$$

We have so far found a strict lower bound on the distortion in (6) and an upper bound in (8) using the best linear scheme. However we still do not know how good linear policies are? In the following we show that the linear policies are person-by-personal optimal, however they do not guarantee global optimality.

3.2. Person-by-person optimality of linear policies

Let us fix the source encoder to be linear. Given a linear and memoryless policy at the source encoder, we now find an optimal relaying policy which minimizes the distortion $\mathbb{E}[(X_n - E[X_n|R^n])^2]$ suffered at time n. We can rewrite the distortion suffered at time n as

$$\begin{aligned} &\mathbb{E}[(X_n - E[X_n|R^n])^2] \\ &\overset{(a)}{=} \mathbb{E}\left[(X_n - E[X_n|Y^n])^2\right] + \mathbb{E}\left[(E[X_n|Y^n] - E[X_n|R^n])^2\right] \\ &\overset{(b)}{=} \mathbb{E}\left[(X_n - c_n Y_n)^2\right] + \mathbb{E}\left[(c_n Y_n - E[X_n|R^n])^2\right], \end{aligned} \tag{9}$$

where (a) follows from $X_n - Y^n - R^n$ and $\mathbb{E}[(X_n - E[X_n|Y^n])(E[X_n|Y^n] - E[X_n|R^n])] = 0$ (by the orthogonality principle); and (b) follows from the fact that the source encoder is linear and memoryless and the MMSE estimation of a Gaussian variable is linear, i.e. $E[X_n|Y^n] = c_n Y_n$, where c_n is a scalar. According to (9), the optimal relaying policy is the one which minimizes $E[(c_n Y_n - E[X_n|R^n])^2]$, since the first term in the distortion function is independent of the relaying policy. This problem was studied in Bansal and Başar (1987), from which it follows that an optimal relay encoding policy is linear and memoryless if we fix the source encoder to be linear memoryless. This observation can also be made from Ayanoglu and Gray (1987), Dobrushin and Tsybakov (1962), Fine (1965), Gastpar (2008), Gomadam and Jafar (2007). Now if we fix the relay encoder policy to be linear and memoryless, one can observe that the problem becomes equivalent to the transmission of a Gaussian source over a point to point Gaussian channel subject to an average power constraint, for which it is well-known that linear (memoryless) encoding is optimal in the sense of minimizing mean-squared distortion (Berger, 1971; Goblick, 1965). Hence, linear encoding policies are person-by-person optimal.

3.3. Concavity of the team problem

In a decentralized team optimization problem person-by-person optimal solutions are globally optimal if the cost function is convex in the policies of the decision makers and the cost function satisfies certain differentiability conditions in the policies (Radner, 1962). Let P be an observation channel from the input variable X at source encoder to the channel output variable R such that $P(\cdot|x)$ is a probability measure on the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ on $\mathbb{R}$ for every $x \in \mathbb{R}$, and $P(A|\cdot) : \mathbb{R} \mapsto [0 : 1]$ is a Borel measurable function for every $A \in \mathcal{B}(\mathbb{R})$. Similarly we define P_1 as an observation channel from the variable X to the variable Y, and P_2 as an observation channel from the variable Y to the variable R. From Yüksel and Başar (2013), Yüksel and Linder (2011, 2012) it follows that the distortion in (3) is concave in the joint observation channel $P(A|x) = \int_{\mathbb{R}} P_2(A|y) P_1(dy|x)$ for every $A \in \mathcal{B}(\mathbb{R})$. If the encoding policies are viewed as stochastic kernels, then the individual observation channels P_1 and P_2 are given by convolutions of Gaussian distributions with the encoding policies, i.e., $P_1 = P_{S_e|X} * \mathcal{N}(0, N_r)$ and $P_2 = P_{S_r|Y} * \mathcal{N}(0, N_d)$. Since the distortion is concave in the joint channel and the individual channels are affine in the source and the relay encoding policies, the distortion is concave in the encoding policies $(P_{S_e|X}, P_{S_r|Y})$ and the optimal policies have to be deterministic if they exist. This implies that the person-by-person optimal encoding policies do not guarantee global optimality when policies are viewed as stochastic kernels (Wu & Verdú, 2011; Yüksel & Başar, 2013; Yüksel & Linder, 2011, 2012).

4. Counterexample: non-linear policies

In this section, we provide a simple counterexample to show that linear policies are not optimal for causal transmission of a Gaussian source over the given two-hop relay channel. Consider the following time invariant policies at the source encoder and the relay encoder respectively:

$$f_1(x) = \begin{Bmatrix} a, & \text{for } x > m_1 \\ 0, & \text{for } |x| \le m_1 \\ -a, & \text{for } x < -m_1 \end{Bmatrix}, \tag{10}$$

$$f_2(y) = \begin{Bmatrix} b, & \text{for } y > m_2 \\ 0, & \text{for } |y| \le m_2 \\ -b, & \text{for } y < -m_2 \end{Bmatrix}, \tag{11}$$

with $a, b \in \mathbb{R}_+$. In (10) and (11) we have dropped the index n for the sake of simplicity without any loss as we are considering time invariant policies. According to these policies, the signals observed at the relay and the destination are respectively given by

$$Y = \begin{Bmatrix} a + Z_r, & \text{for } X > m_1 \\ Z_r, & \text{for } |X| \le m_1 \\ -a + Z_r, & \text{for } X < -m_1 \end{Bmatrix}, \tag{12}$$

$$R = \begin{Bmatrix} b + Z_d, & \text{for } Y > m_2 \\ Z_d, & \text{for } |Y| \le m_2 \\ -b + Z_d, & \text{for } Y < -m_2 \end{Bmatrix}. \tag{13}$$

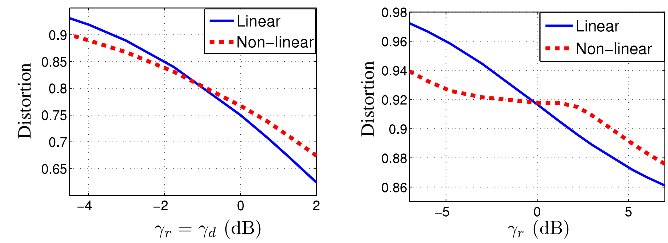
The policies in (10) and (11) have to satisfy the transmit power constraints in (1) and (2). In Appendix A we have obtained conditions on $a, b \in \mathbb{R}_+$ to ensure these power constraints, which are given by (A.1) and (A.4) respectively. For these non-linear policies, the MMSE decoder $g(R)$ and the corresponding distortion D_{NL} are given in (B.7) and (B.8). We can numerically compute D_{NL} using (B.8), (A.1), and (A.4) for any fixed values of the parameters $\{\sigma_x^2, P_S, P_R, N_d, N_r, m_1, m_2\}$. We now give two examples to demonstrate that the proposed simple non-linear scheme can outperform the best linear scheme. In the following examples we fix the values of the system parameters and then numerically compute the distortion for non-linear and linear policies according to (B.8) and (8) respectively. We also evaluate the lower bound in (6), however the reader should keep in mind that the bound is not tight in general.

Example 1. Fixing $\sigma_x^2 = P_S = P_R = 1$, $N_r = N_d = 4$, $m_1 = 2.45$, $m_2 = 6.84$, $a = 8.36$, and $b = 9.25$ we get: $D_{NL} = 0.926$, $D_L^\star = 0.96$, and $D = 0.8$.

Example 2. Fixing $\sigma_x^2 = P_S = P_R = 1$, $N_r = N_d = 10$, $m_1 = 2.85$, $m_2 = 12.05$, $a = 15.12$, and $b = 16.25$ we get: $D_{NL} = 0.964$, $D_L^\star = 0.992$, and $D = 0.909$.

These examples validate that linear policies are not optimal in general for the given two-hop relay channel when the source and the relay node have individual power constraints. Let us now consider a total power, $\mathbb{E}[S_{e,n}^2] + \mathbb{E}[S_{r,n}^2] = P$. It can be easily shown that the distortion is minimized for the linear policies by an equal power allocation $\mathbb{E}[S_{e,n}^2] = \mathbb{E}[S_{r,n}^2] = \frac{P}{2}$ if $N_r = N_d$. In the above two counterexamples equal power constraints and noise variances are imposed on the source and relay nodes. Therefore, linear policies are still sub-optimal even if a total transmit power constraint is imposed on the source and the relay, instead of separate power constraints. Note that for more than one relay nodes linked in cascade (multi-hop channel), the end-to-end distortion can be written as the sum of distortions suffered at each hop since the source input is memoryless and by applying the orthogonality property, as we did for the two-hop case in (9). Therefore linear policies are sub-optimal for multi-hop relaying in general.

The proposed non-linear scheme is not always better than the optimal linear scheme as demonstrated in Fig. 2, where we have plotted distortion achieved with the non-linear and the optimal linear schemes as functions of signal-to-noise ratios for some fixed parameters. The non-linear scheme outperforms the linear scheme in low SNR regions, however there might exist better non-linear strategies which may outperform the linear strategy also in high SNR regions. When the channels are very noisy, one intuition on why the proposed non-linear strategy is superior may be that it does not amplify the large values of channel noise at its input unlike the linear (amplify-and-forward) strategy. We note that, in Lipsa and Martins (2011) a two-level quantizer was used when the number of relays were greater than two. In our setting the result also holds for a single relay, which generalizes and implies the results of Lipsa and Martins (2011). The reason for selecting symmetric quantizers is due to the fact that symmetry in distribution is preserved when symmetric functions are applied to sources with symmetric distributions. Moreover with centering the quantizer at zero, the encoders can utilize the available transmit power in an efficient way by transmitting signals with power equal to zero more often.



(a) $\sigma_x^2 = P_S = P_R = 1$. (b) $\sigma_x^2 = P_S = P_R = 1, N_d = 5$.

Fig. 2. Comparison of linear and non-linear schemes.

5. Conclusion

We studied the problem of mean-square estimation of a Gaussian source over a two-hop Gaussian relay channel with average transmit power constraints. A lower bound on the distortion was derived. We observed that the distortion bounds obtained using cut-set arguments are not tight in general for sensor networks if we restrict policies to be memoryless. Further it was shown that linear policies are person-by-person optimal over the given two-hop relay channel. However person-by-person optimality of the linear policies does not guarantee global optimality due to the concavity property of the distortion function in the observation channel. A simple three-level function was shown to outperform the best linear scheme in some cases, thus validating that linear policies are not optimal in general. This observation is in accordance with the previously known results for non-classical information structures (Ho, 1980). We wish to identify necessary and sufficient conditions for optimal schemes using variational methods as future work. Some recent related results on MMSE, linearity of optimal estimation and the Witsenhausen's counterexample can be found in Akyol, Viswanatha, and Rose (2012), Wu and Verdú (2010, 2011).

Appendix A. Transmit power constraints

The parameter $a \in \mathbb{R}_+$ in (10) is chosen such that

$$\begin{aligned} P_S &\ge \mathbb{E}\left[f_1^2(X)\right] \\ &\overset{(a)}{=} a^2 \left(\int_{-\infty}^{-m_1} p(x)dx + \int_{m_1}^{\infty} p(x)dx \right) \\ &\overset{(b)}{=} 2a^2 Q\left(\frac{m_1}{\sigma_x}\right) \Rightarrow a \le \sqrt{\frac{P_S}{2Q\left(\frac{m_1}{\sigma_x}\right)}}, \end{aligned} \tag{A.1}$$

where (a) follows from (10); (b) follows from $p(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{x^2}{2\sigma_x^2}}$, $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{\tau^2}{2}} d\tau = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-x} e^{-\frac{\tau^2}{2}} d\tau$. We now compute

$p(y)$ to find the condition on $b \in \mathbb{R}_+$ in (11) which ensures $\mathbb{E}\left[S_r^2\right] \leq P_R$. From (12) we have

$$p(y|x) = \left\{ \begin{array}{ll} \frac{1}{\sqrt{2\pi N_r}} e^{-\frac{(y-a)^2}{2N_r}}, & \text{if } x > m_1 \\ \frac{1}{\sqrt{2\pi N_r}} e^{-\frac{y^2}{2N_r}}, & \text{if } |x| \leq m_1 \\ \frac{1}{\sqrt{2\pi N_r}} e^{-\frac{(y+a)^2}{2N_r}}, & \text{if } x < -m_1 \end{array} \right\}. \quad \text{(A.2)}$$

The marginal pdf $p(y)$ can now be computed as

$$\begin{aligned} p(y) &= \int_{\mathbb{R}} p(y|x)p(x)dx \\ &= \frac{1}{\sqrt{2\pi N_r}} \left(e^{\frac{-(y+a)^2}{2N_r}} \int_{-\infty}^{-m_1} p(x)dx \right. \\ &\quad \left. + e^{\frac{-y^2}{2N_r}} \int_{-m_1}^{m_1} p(x)dx + e^{\frac{-(y-a)^2}{2N_r}} \int_{m_1}^{\infty} p(x)dx \right) \\ &= \frac{1}{\sqrt{2\pi N_r}} \left[\left(e^{\frac{-(y+a)^2}{2N_r}} + e^{\frac{-(y-a)^2}{2N_r}} \right) Q\left(\frac{m_1}{\sigma_x}\right) \right. \\ &\quad \left. + e^{\frac{-y^2}{2N_r}} \left(1 - 2Q\left(\frac{m_1}{\sigma_x}\right)\right)\right]. \end{aligned} \quad \text{(A.3)}$$

The condition on b is obtained as follows:

$$\begin{aligned} P_R \geq\ & \mathbb{E}\left[f_2^2(Y)\right] \\ \stackrel{(a)}{=}\ & b^2 \left(\int_{-\infty}^{-m_2} p(y)dy + \int_{m_2}^{\infty} p(y)dy \right) \\ \stackrel{(b)}{=}\ & 2b^2 \int_{-\infty}^{-m_2} p(y)dy \\ \stackrel{(c)}{=}\ & 2b^2 \left[Q\left(\frac{m_1}{\sigma_x}\right) \left\{ \frac{1}{\sqrt{2\pi N_r}} \int_{-\infty}^{-m_2} e^{\frac{-(y+a)^2}{2N_r}} dy \right.\right. \\ & \left. + \frac{1}{\sqrt{2\pi N_r}} \int_{-\infty}^{-m_2} e^{\frac{-(y-a)^2}{2N_r}} dy \right\} \\ & \left. + \left(1 - 2Q\left(\frac{m_1}{\sigma_x}\right)\right) \left\{ \frac{1}{\sqrt{2\pi N_r}} \int_{-\infty}^{-m_2} e^{\frac{-y^2}{2N_r}} dy \right\} \right] \\ \stackrel{(d)}{=}\ & 2b^2 \left[Q\left(\frac{m_1}{\sigma_x}\right) \left(Q\left(\frac{m_2 - a}{\sqrt{N_r}}\right) + Q\left(\frac{m_2 + a}{\sqrt{N_r}}\right)\right) \right. \\ & \left. + Q\left(\frac{m_2}{\sqrt{N_r}}\right) \left(1 - 2Q\left(\frac{m_1}{\sigma_x}\right)\right)\right] \\ \stackrel{(e)}{=:}\ & 2b^2 \kappa(m_1, m_2, a, \sigma_x, N_r) \\ \Rightarrow b \leq\ & \sqrt{\frac{P_R}{2\kappa(m_1, m_2, a, \sigma_x, N_r)}}, \end{aligned} \quad \text{(A.4)}$$

where (a) follows from (11); (b) follows from symmetry of $p(y)$ around origin; (c) follows by substituting $p(y)$ from (A.3); (d) follows by the definition of $Q(\cdot)$ function; and (e) follows by defining $\kappa(m_1, m_2, a, \sigma_x, N_r)$.

Appendix B. Distortion calculation

Since we have the following Markov chain $X - Y - R$, $p(x, r)$ is given by

$$p(x, r) = \int_{\mathbb{R}} p(r|y)p(y|x)p(x)dy, \quad \text{(B.1)}$$

where $p(r|y)$ follows from (13), that is

$$p(r|y) = \left\{ \begin{array}{ll} \frac{1}{\sqrt{2\pi N_d}} e^{-\frac{(r-b)^2}{2N_d}}, & \text{if } y > m_2 \\ \frac{1}{\sqrt{2\pi N_d}} e^{-\frac{r^2}{2N_d}}, & \text{if } |y| \leq m_2 \\ \frac{1}{\sqrt{2\pi N_d}} e^{-\frac{(r+b)^2}{2N_d}}, & \text{if } y < -m_2 \end{array} \right\}. \quad \text{(B.2)}$$

From (A.2) we see that $p(y|x)$ is defined over the three disjoint intervals of x (i.e., $x\langle m_1$, $|x| \leq m_1$, $x\rangle m_1$). Due to this, the joint pdf $p(x, r)$ is also defined over these three intervals. For the interval $x < -m_1$,

$$\begin{aligned} p(x, r) &\stackrel{(a)}{=} \frac{p(x)}{\sqrt{2\pi N_d}} \left[\frac{e^{\frac{-(r+b)^2}{2N_d}}}{\sqrt{2\pi N_r}} \int_{-\infty}^{-m_2} e^{\frac{-(y+a)^2}{2N_r}} dy \right. \\ &\quad + \frac{e^{\frac{-r^2}{2N_d}}}{\sqrt{2\pi N_r}} \int_{-m_2}^{m_2} e^{\frac{-(y+a)^2}{2N_r}} dy \\ &\quad \left. + \frac{e^{\frac{-(r-b)^2}{2N_d}}}{\sqrt{2\pi N_r}} \int_{m_2}^{\infty} e^{\frac{-(y+a)^2}{2N_r}} dy \right] \\ &= \frac{p(x)}{\sqrt{2\pi N_d}} \left[e^{\frac{-(r+b)^2}{2N_d}} Q\left(\frac{m_2 - a}{\sqrt{N_r}}\right) \right. \\ &\quad + e^{\frac{-(r-b)^2}{2N_d}} Q\left(\frac{m_2 + a}{\sqrt{N_r}}\right) \\ &\quad \left. + e^{\frac{-r^2}{2N_d}} \left\{ 1 - Q\left(\frac{m_2 - a}{\sqrt{N_r}}\right) - Q\left(\frac{m_2 + a}{\sqrt{N_r}}\right)\right\}\right] \\ &\stackrel{(b)}{=:} p(x) l_1(r), \end{aligned} \quad \text{(B.3)}$$

where (a) follows from (B.1) and (A.2) and (B.2); and (b) follows by defining $l_1(r)$. Similarly for $|x| \leq m_1$,

$$\begin{aligned} p(x, r) &= \frac{p(x)}{\sqrt{2\pi N_d}} \left[e^{\frac{-r^2}{2N_d}} + \left\{ e^{\frac{-(r+b)^2}{2N_d}} \right.\right. \\ &\quad \left.\left. + e^{\frac{-(r-b)^2}{2N_d}} - 2e^{\frac{-r^2}{2N_d}} \right\} Q\left(\frac{m_2}{\sqrt{N_r}}\right)\right] \\ &=: p(x) l_2(r), \end{aligned} \quad \text{(B.4)}$$

and for $x > m_1$,

$$\begin{aligned} p(x, r) &= \frac{p(x)}{\sqrt{2\pi N_d}} \left[e^{\frac{-(r+b)^2}{2N_d}} Q\left(\frac{m_2 + a}{\sqrt{N_r}}\right) \right. \\ &\quad + e^{\frac{-r^2}{2N_d}} \left\{ 1 - Q\left(\frac{m_2 - a}{\sqrt{N_r}}\right) - Q\left(\frac{m_2 + a}{\sqrt{N_r}}\right)\right\} \\ &\quad \left. + e^{\frac{-(r-b)^2}{2N_d}} Q\left(\frac{m_2 - a}{\sqrt{N_r}}\right)\right] =: p(x) l_3(r). \end{aligned} \quad \text{(B.5)}$$

From (B.3), (B.4), and (B.5), we compute

$$\begin{aligned} p(r) &= \int_{\mathbb{R}} p(x, r)dx \\ &= l_1(r) \int_{-\infty}^{-m_1} p(x)dx + l_2(r) \int_{-m_1}^{m_1} p(x)dx \\ &\quad + l_3(r) \int_{m_1}^{\infty} p(x)dx \\ &= (l_1(r) + l_3(r))\, Q\left(\frac{m_1}{\sigma_x}\right) + l_2(r) \left(1 - 2Q\left(\frac{m_1}{\sigma_x}\right)\right). \end{aligned} \quad \text{(B.6)}$$

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The MMSE estimator can now be computed using (B.3), (B.4), (B.5), and (B.6) as follows.

$$\begin{aligned}
&\mathbb{E}[X|R=r] \\
&= \frac{1}{p(r)} \int x p(x,r) dx \\
&= \frac{1}{p(r)} \left(l_1(r) \int_{-\infty}^{-m_1} x p(x) dx \right. \\
&\quad \left. + l_2(r) \int_{-m_1}^{m_1} x p(x) dx + l_3(r) \int_{m_1}^{\infty} x p(x) dx \right) \\
&\overset{(a)}{=} \frac{1}{p(r)} (l_3(r) - l_1(r)) \int_{m_1}^{\infty} x p(x) dx \\
&= \frac{1}{p(r)} (l_3(r) - l_1(r)) \sqrt{\frac{\sigma_x^2}{2\pi}} \exp\left(-\frac{{m_1}^2}{2\sigma_x^2} \right) \\
&=: g(r), \qquad \text{(B.7)}
\end{aligned}$$

where (a) follows from $\int_{-m_1}^{m_1} x p(x) dx = 0$. The associated mean-squared error is

$$\begin{aligned}
D_{NL} &:= \int_{\mathbb{R}^2} (x - g(r))^2 p(x,r) d(x,r) \\
&= \int_{-\infty}^{\infty} \left(l_1(r) \int_{-\infty}^{-m_1} (x - g(r))^2 p(x) dx \right. \\
&\quad + l_2(r) \int_{-m_1}^{m_1} (x - g(r))^2 p(x) dx \\
&\quad \left. + l_3(r) \int_{m_1}^{\infty} (x - g(r))^2 p(x) dx \right) dr. \qquad \text{(B.8)}
\end{aligned}$$

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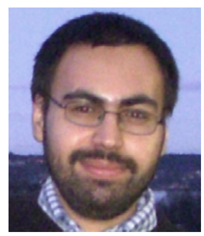

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